04 April 2018 08:52

Phase Response of LTI Systems

Significance of Phase Response

Example 1

$$H(e^{j\omega}) = e^{-j\omega N_d}$$

e

$$|H(e^{j\omega})| = 1 \quad \forall\, \omega$$

$$x(n) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow y(n)$$

$$Y(e^{j\omega}) = e^{-j\omega N_d}\, X(e^{j\omega})$$

If N_d is an integer

$$y(n) = x(n - N_d)$$

(delay property of DTFT)

If N_d is not an integer

Say $x(n)$ is obtained from periodic Sampling $x_c(t)$ sampling at interval T

$$y(n) = x_c(t - N_d T) \text{ sampled}$$

$= x_c(nT - N_d T)$ at $t = nT$

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(Nd integer)

$$x(n) = e^{jw_0 n}$$

$$y(n) = e^{jw_0(n-Nd)}$$

Any freq w_0 gets delayed by Nd Samples

Example 2

$$H(e^{jw}) = e^{-j|w|Nd}$$

$$= \begin{cases} e^{-jwNd}, & ; w>0 \\ e^{+jwNd} & ; w<0 \end{cases}$$

$$x(n) \longrightarrow \boxed{H(e^{jw})} \longrightarrow y(n)$$

Input $w_0 > 0$ output (delay of +Nd)

+ve freq $x_1(n) = e^{jw_0 n}$ $y_1(n) = e^{jw_0(n-Nd)}$

-ve freq. $x_2(n) = e^{-jw_0 n}$ $y_2(n) = e^{-jw_0(n+Nd)}$

delay of (-Nd)

Roughly Speaking, Phase response characterizes the delay introduced by the system for different frequencies.

Definitions

$H(e^{j\omega}) \rightarrow$ frequency response

$$\left.\begin{aligned} \angle H(e^{j\omega}) &= \arg(H(e^{j\omega})) \\ &= \phi(\omega) \end{aligned}\right\} \text{phase response}$$

Group delay response (in number of Samples)

$$\tau(\omega) = -\frac{d}{d\omega}\phi(\omega)$$

(-ve of derivative of phase response)

For Example 1 : $H(e^{j\omega}) = e^{-j\omega N_d}$

$$\phi(\omega) = -\omega N_d$$

$$\tau(\omega) = N_d, \ \forall \omega$$

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Example 2:

$$\phi(\omega) = \begin{cases} -\omega N_d & ; \omega > 0 \\ \omega N_d & ; \omega < 0 \end{cases}$$

$$\tau(\omega) = \begin{cases} N_d & ; \omega > 0 \\ -N_d & ; \omega < 0 \end{cases}$$

x ——————→

Recall phase response

$$\phi(\omega) = \arg(H(e^{j\omega}))$$

↓

Continuous phase function

arg (.) to denote continuous phase response

Wrapped Phase:

$ARG(H(e^{j\omega})) \rightarrow$ wrapped phase response

↓ where we restrict the angle to be within $-\pi$ to π

wrapped phase can have jumps of height 2π (jump from $-\pi$ to π & vice versa)

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We can write

$$\arg(H(e^{j\omega})) = ARG(H(e^{j\omega})) + 2\pi k(\omega)$$

$k(\omega)$ is an integer depending on ω

Note. $\tau(\omega)$ is obtained by taking derivative of continuous phase $\phi(\omega)$ function $\arg(H(e^{j\omega}))$

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Phase Response of Rational Transfer function

$$H(z) = \frac{b_0 \prod_{k=1}^{M} (1 - q_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}$$

$q_1, q_2, \ldots, q_M$ are zeros

$p_1, p_2, \ldots, p_N$ are poles

Now,

$$\phi(\omega) = \angle H(e^{j\omega})$$

$$= \arg H(e^{j\omega})$$

$$= \arg \left\{ \frac{b_0 \prod_{k=1}^{M} (1 - q_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - p_k e^{-j\omega})} \right\}$$

$$= \arg(b_0) + \sum_{k=1}^{M} \arg(1 - q_k e^{-j\omega})$$

$$- \sum_{k=1}^{N} \arg(1 - p_k e^{-j\omega})$$

- Each term in summation is of form $\arg(1 - a e^{-j\omega})$
- Effect of pole is same as zero except for negative sign (reversal of sign)

First order zero $\theta = 0$

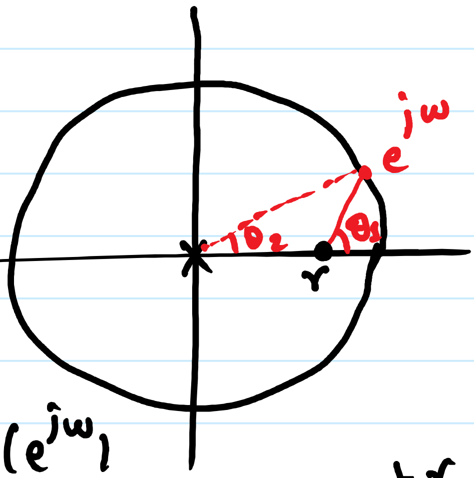
$$H(z) = 1 - r z^{-1}$$

$$H(e^{j\omega}) = 1 - r e^{-j\omega}$$

$$\phi(\omega) = \arg(1 - r e^{-j\omega})$$

$$= \arg(e^{j\omega} - r) - \arg(e^{j\omega})$$

$$= \theta_1 - \theta_2$$



(zero at r
pole at origin)

Geometric Interpretation:

θ_1 = angle of line joining $e^{j\omega}$ & r

θ_2 = angle of line joining $e^{j\omega}$ & 0

~~Non~~ Trivial poles/zeros at origin will have an impact on phase response.

$$H(z) = 1 - rz^{-1}$$

$$h(n) = \{ \underset{\substack{\uparrow \\ n=0}}{1}, -r \}$$

$h(n)$ is real valued

so $|H(e^{j\omega})| \rightarrow$ even function

$\angle H(e^{j\omega}) \rightarrow$ odd function

$\longrightarrow$

Consider few values of ω

$\omega = 0$; $\theta_1 = \theta_2 = 0$; $\phi(\omega) = 0$

$\omega = \pi$; $\theta_1 = \theta_2 = \pi$; $\phi(\omega) = 0$

When $\omega > 0$ (for small values of ω)

θ_1 increases faster than θ_2

$\theta_1 - \theta_2 > 0$

As ω increases $\theta_1 - \theta_2$ increases and reaches max at some point beyond which it decreases back to 0

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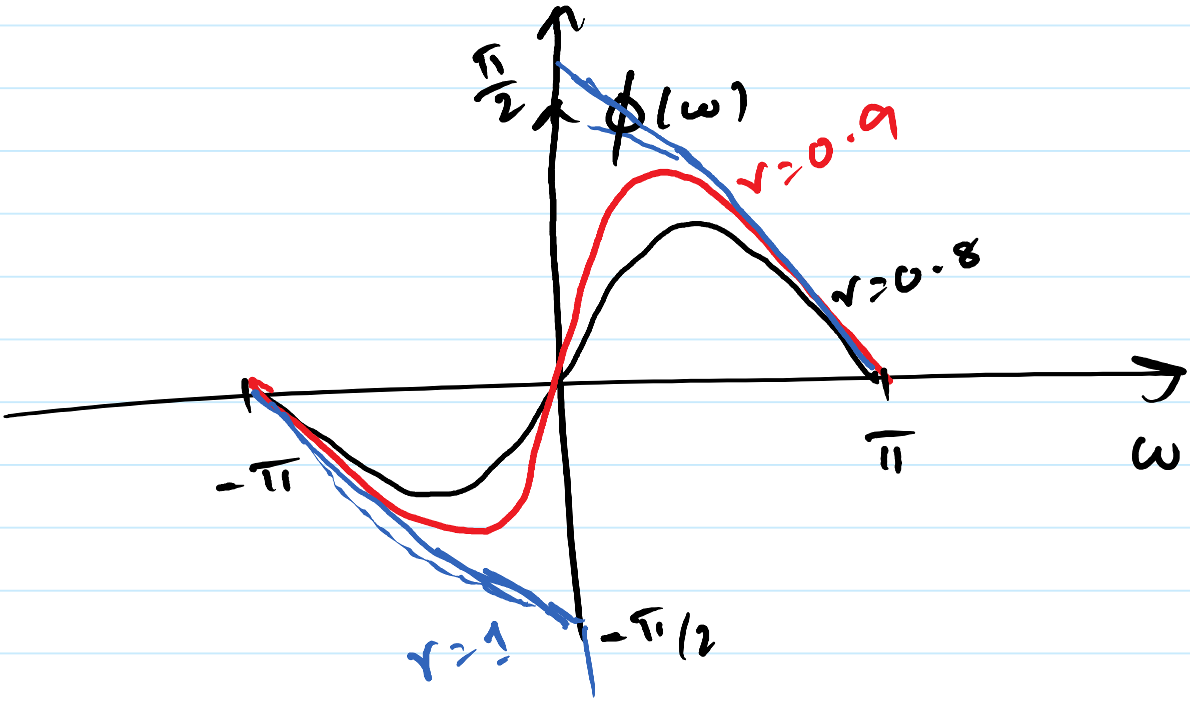
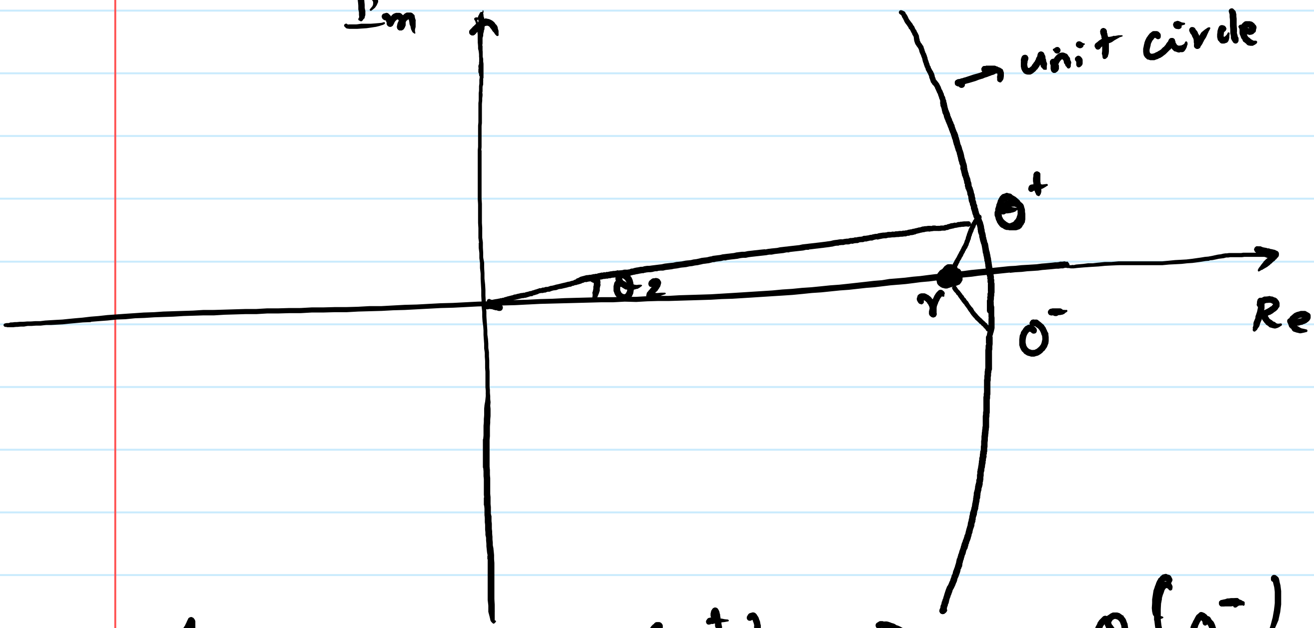
Plot of $\phi(\omega)$ Vs ω



As $r \to 1$



As $r \to 1$, $\theta_1(0^+) \to \pi/2$ $\theta_1(0^-) \to -\pi/2$

$\theta_2(0^+) \to 0$ $\theta_2(0^-) \to 0$

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$$\phi(0^+) - \phi(0^-) = \pi$$

As $r \to 1$, there is a <u>phase jump</u> of π as we cross the <u>zero</u>

For $r = 1$,

$$\phi(\omega) = \tan^{-1}\left(\frac{\sin\omega}{1-\cos\omega}\right)$$

$$= \arg(1 - e^{-j\omega})$$

$\omega > 0$

$$= \tan^{-1}\left(\tan\left(\pi/2 - \omega/2\right)\right)$$

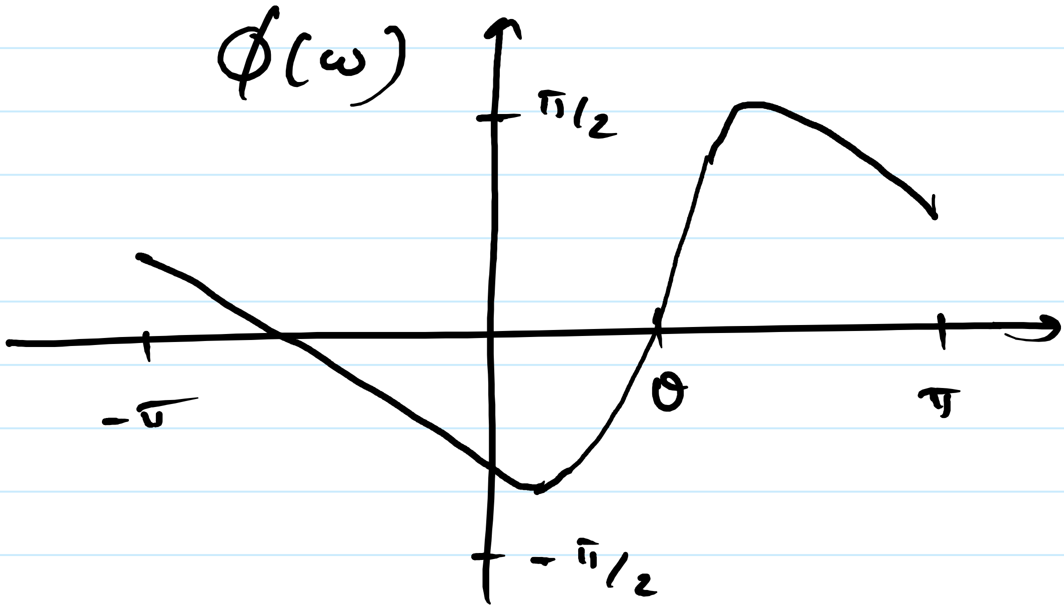
$$= \pi/2 - \omega/2 \quad \text{if } \omega > 0$$

$$\phi(\omega) = \begin{cases} \pi/2 - \omega/2 & \text{if } \omega > 0 \\ -\pi/2 + \omega/2 & \text{if } \omega < 0 \end{cases}$$

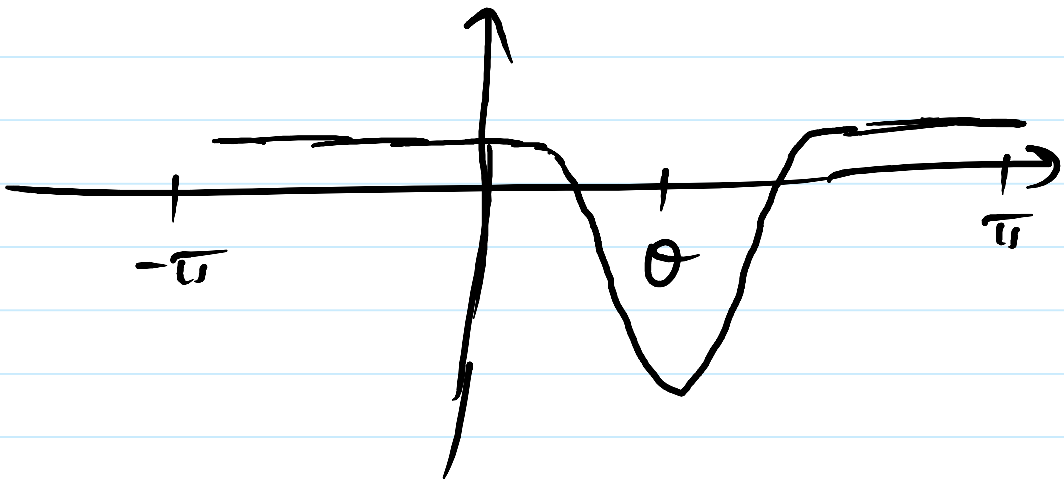
$$\tau(\omega) = 1/2 \quad \text{if } \omega \neq 0$$

For the general case when zero @ $re^{j\theta}$

$$H(e^{j\omega}) = 1 - re^{j\theta}e^{-j\omega}$$



Group delay $\tau(\omega)$



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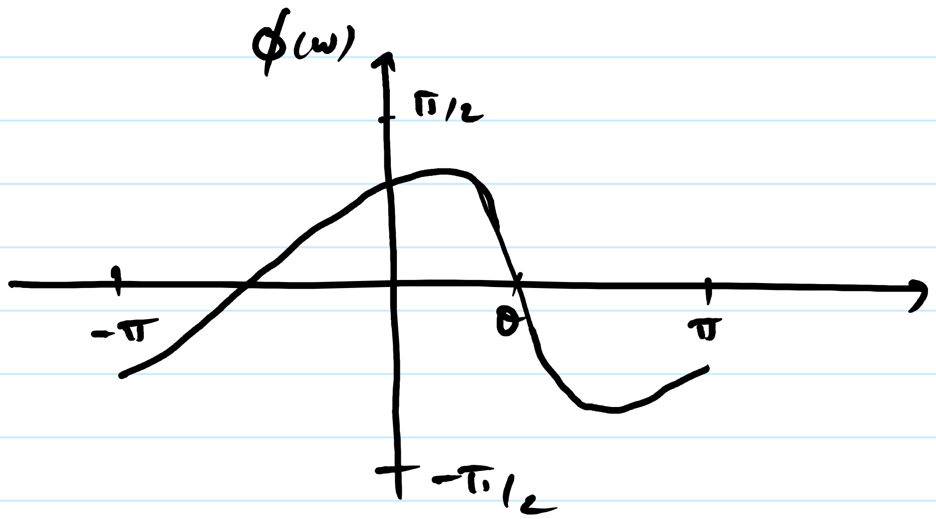
First order pole

$$H(z) = \frac{1}{1 - re^{j\theta} z^{-1}}$$

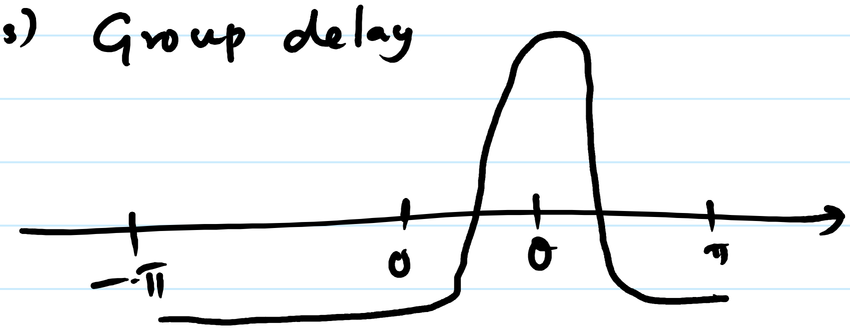
$$\phi(\omega) = \angle H(e^{j\omega})$$

$$= - \arg \{ 1 - re^{j\theta} e^{-j\omega} \}$$

= -ve of first order zero



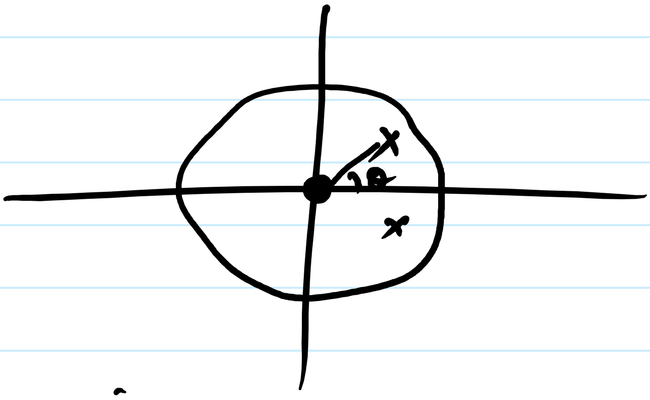
(samples) Group delay



Second Order System

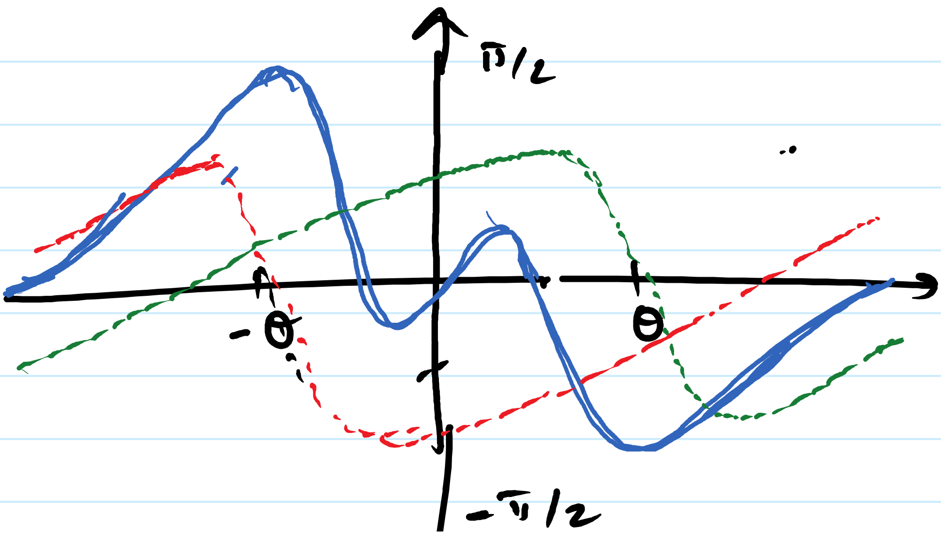
Resonator

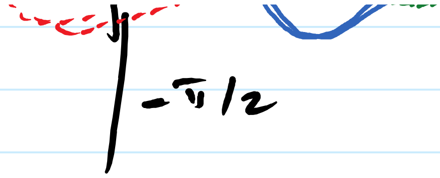
$$H(z) = \frac{1}{(1 - re^{j\theta} z^{-1})(1 - re^{-j\theta} z^{-1})}$$



$$\phi(\omega) = \angle H(e^{j\omega})$$

$$\left.\begin{aligned} &= -\arg\{1 - re^{j\theta} e^{-j\omega}\} \\ &\quad -\arg\{1 - re^{-j\theta} e^{-j\omega}\} \end{aligned}\right\} \text{sum of first order respon}$$




−π/2

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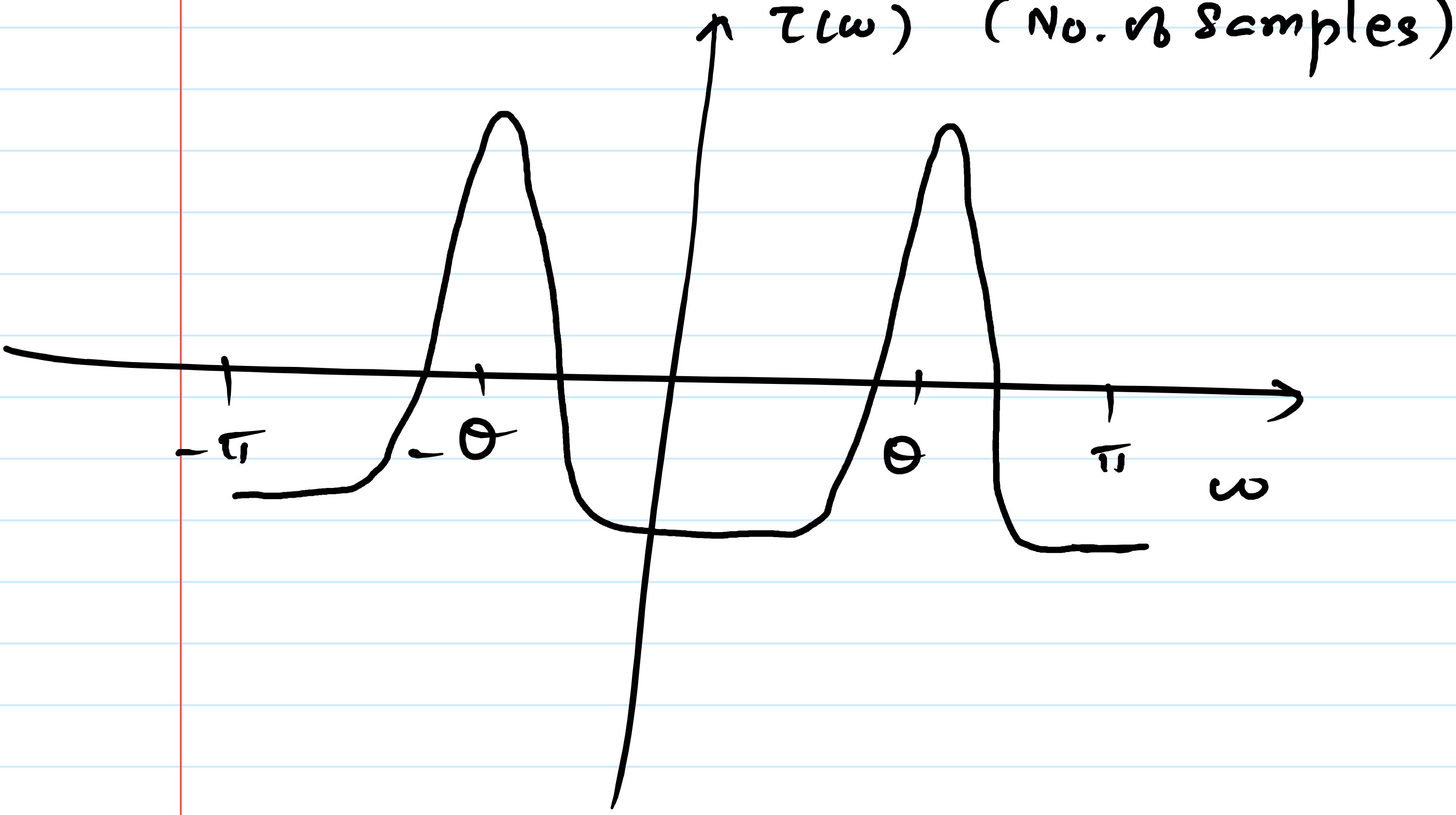
Group Delay Response



Generalized Linear Phase Systems

- Generalized linear phase (GLP)

Phase Response $\phi(\omega) = \alpha\omega + \beta$

α, β are constants

- Group delay Response of GLP

$$\tau(\omega) = -\frac{d}{d\omega}\phi(\omega)$$

$$= -\alpha \quad , \text{ for all } \omega$$

- GLP systems have <u>same delay</u> for all frequencies

- Constant Group delay is desirable in many practical systems

Frequency response $H(e^{j\omega})$ for a GLP system can be written as

$$H(e^{j\omega}) = A(e^{j\omega})\, e^{+j(\omega\alpha+\beta)}$$

$A(e^{j\omega})$ is real valued function

(can take positive and negative values)

$A(e^{j\omega})$ is called amplitude response of system

Questions:

1. How to design systems which has GLP?

2. what does the impulse response of GLP satisfy? (conditions)

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(real valued)

Impulse response of GLP

Say $h(n)$ is real valued

suppose $H(e^{j\omega}) = A(e^{j\omega})\, e^{j(\omega\alpha+\beta)}$

Now, using Euler's formula

(a) — $H(e^{j\omega}) = A(e^{j\omega}) \cos(\omega\alpha+\beta) + j\, A(e^{j\omega}) \sin(\omega\alpha+\beta)$

From DTFT formula,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

(b) — $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) \cos\omega n - j \sum_{n=-\infty}^{\infty} h(n) \sin(\omega n)$

Comparing the phase of (a) & (b)

$$\tan(\omega\alpha+\beta) = \frac{\sin(\omega\alpha+\beta)}{\cos(\omega\alpha+\beta)} = \frac{-\sum_n h(n) \sin \omega n}{\sum_n h(n) \cos \omega n}$$

Cross multiplying & using trignometry formulas

$$\boxed{\sum_{n=-\infty}^{\infty} h(n) \sin[\omega(n+\alpha)+\beta] = 0}$$

$$\sum_{n=-\infty} h(n) \sin[\omega(n+\alpha) \pm \beta] = 0$$

for all ω

① One class of GLP systems (even symmetry)

Set these values
$$\begin{cases} \beta = 0 \\ -2\alpha = M \quad \text{(integer)} \end{cases}$$

$$\phi(\omega) = \omega \frac{M}{2}$$

$$H(e^{j\omega}) = A(e^{j\omega})\, e^{-j\omega \frac{M}{2}}$$

Required condition on impulse response

$$(*) \;—\; \sum_{n=-\infty}^{\infty} h(n) \sin\left[\omega\left[n - \frac{M}{2}\right]\right] = 0$$

for all ω

Consider the "even symmetry"

$$h(n) = h(M-n)$$

In this case, consider terms

$n = n_0$ and $n = M - n_0$

$$h(n_0) \sin(\omega(n_0 - M/2))$$
$$h(M-n_0) \sin(\omega(M - n_0 - M/2))$$

negative of each other

Terms in (A) can be paired
(n_0 with $M-n_0$)
so that each pair sums to zero

Sufficient condition for GLP

$$\boxed{h(n) = h(M-n)}$$ even symmetry

M is an integer

② Another class of GLP

Set $\beta = \pi/2$

$-2\alpha = M$ (integer)

In this case, the "symmetry"

condition is

$$\boxed{h(n) = -h(M-n)}$$

$\Rightarrow$ odd symmetry

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Causal GLP systems

In addition to symmetry

we also impose causality condition

$$\left\{\begin{array}{l} h(n) = 0 \quad \text{if } n < 0 \\ h(n) = h(M-n) \end{array}\right.$$

together

$$h(n) = \begin{cases} h(M-n) & , \ 0 \le n \le M \\ 0 & ; \ \text{otherwise} \end{cases}$$

even symmetry FIR

Similary, we get odd symmetry FIR

$$h(n) = \begin{cases} -h(M-n) & , \ 0 \le n \le M \\ 0 & ; \ \text{else} \end{cases}$$

Note $\phi(\omega) = \omega\alpha + \beta = -\omega\frac{M}{2} + \beta$

$\tau(\omega) = M/2$

$M =$ even / odd gives has integer / non integer delay

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4 types of FIR GLP

M = odd / even

Symmetry = odd / even

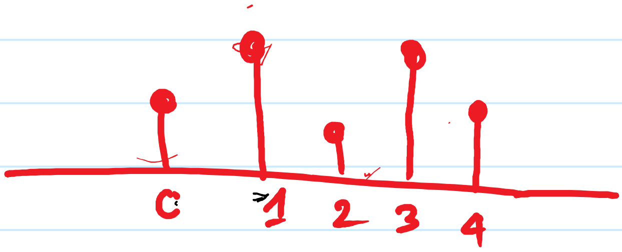
Type I FIR GLP

$$h(n) = h(M-n) \qquad 0 \le n \le M$$

$$M = \text{even}$$

Example

$M = 4$



Now

$$H(e^{j\omega}) = \sum_{n=0}^{M} h(n) e^{-j\omega n}$$

$$= e^{-j\omega \frac{M}{2}} \underbrace{\sum_{k=0}^{M/2} a(k) \cos(\omega k)}_{A(e^{j\omega})}$$

$$a(0) = h(M/2)$$

$$a(k) = 2h(M/2 - k), \quad k = 1, \dots, M/2$$

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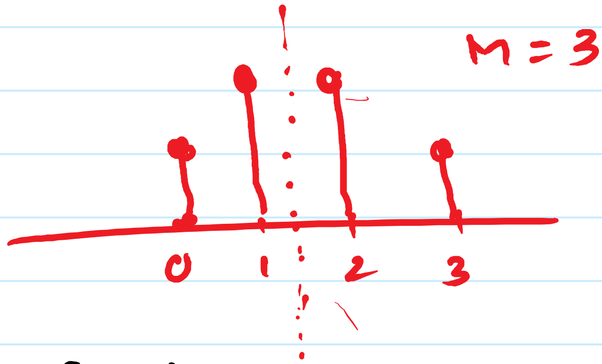
Type II FIR GLP

$$h(n) = h(M-n) \qquad M = \text{odd}$$



$$H(e^{j\omega}) = e^{-j\omega\frac{M}{2}} A(e^{j\omega})$$

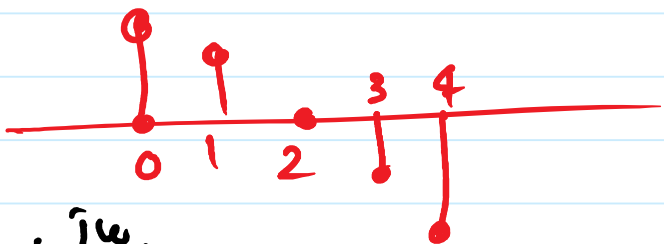
where

$$A(e^{j\omega}) = \sum_{k=1}^{M+1/2} 2h\left(\frac{M+1}{2}-k\right) \cdot \cos(\omega(k-1/2))$$

Type III FIR GLP

$$h(n) = -h(M-n) \qquad M = \text{even}$$

Note $h(M/2) = 0$



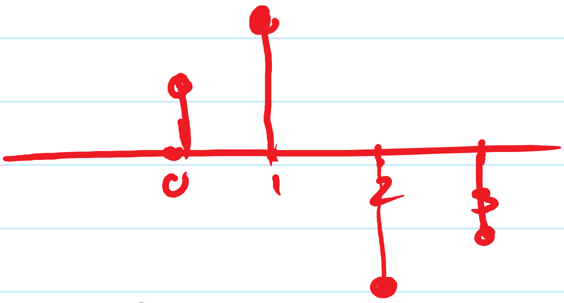
$$H(e^{j\omega}) = j\, e^{-j\omega\frac{M}{2}} A(e^{j\omega})$$

Type 4 GLP

$$h(n) = -h(M-n)$$

M = odd

$M = 3$



$$H(e^{j\omega}) = j\, e^{-j\omega \frac{M}{2}} A(e^{j\omega})$$

All filters have same groupdelay

$$\tau(\omega) = M/2$$

Due to symmetry, there are some constraints on the z – transform

Specifically $H(z)$ at $z=1$, $z=-1$

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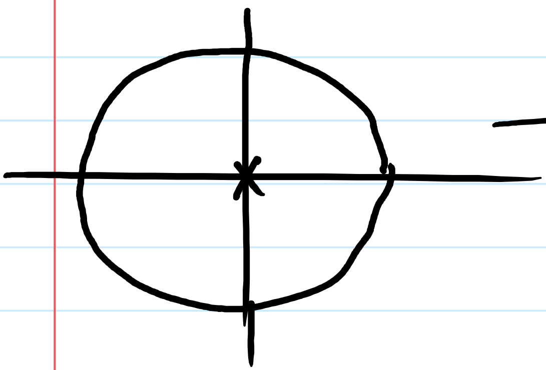
$$H(z)\Big|_{@z=1} = \sum_{n=0}^{M} h(n)$$

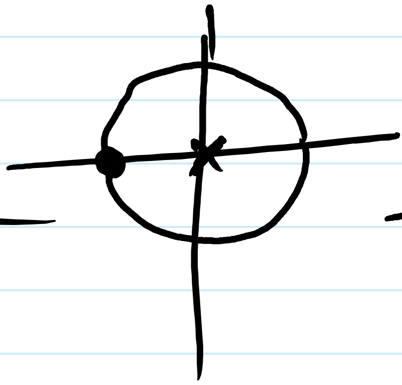
$$H(z)\Big|_{@z=-1} = \sum_{n=0}^{M} h(n)(-1)^n$$

Type I



No constraints on $H(1)$, $H(-1)$

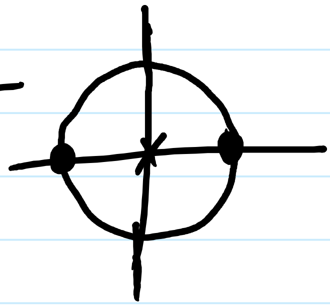
Type II



$H(-1) = 0$

Can't be HPF

Type III



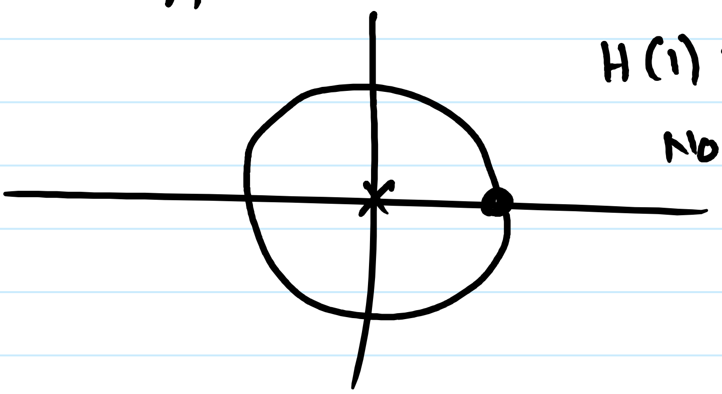
$H(1) = 0$

$H(-1) = 0$

No LPF

No HPF

Type 4



$H(1) = 0$

No LPF

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Additional Constraints

If $h(n)$ ~~is~~ real valued,

$$h(n) = h^*(n)$$

$$\Rightarrow \quad H(z) = H^*(z^*)$$

If z_0 is a zero,

then z_0^* is also a zero

Due to symmetry,

$$h(n) = \pm h(M-n)$$

$$\Rightarrow \quad H(z) = \pm z^{-M} H(z^{-1})$$

If z_0 is a zero,

then $\frac{1}{z_0}$ is also a zero