Freq

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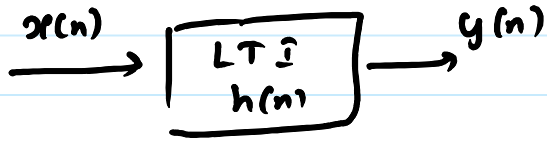
Frequency Domain Analysis of LTI Systems

$h(n)$ impulse response

$H(e^{jw})$ Frequency response



$$y(n) = x(n) * h(n)$$

$$Y(e^{jw}) = X(e^{jw})\, H(e^{jw})$$

Two components

1) Magnitude Response $|H(e^{jw})|$
2) Phase Response $\angle H(e^{jw})$

Will address:

(a) what is the effect of a given response $H(e^{jw})$ on the input?

(b) How to design practical systems with desired response?

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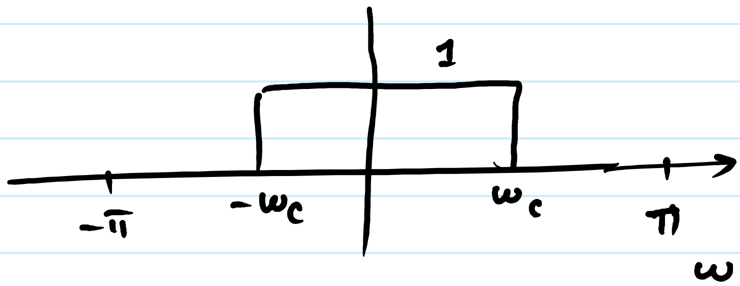
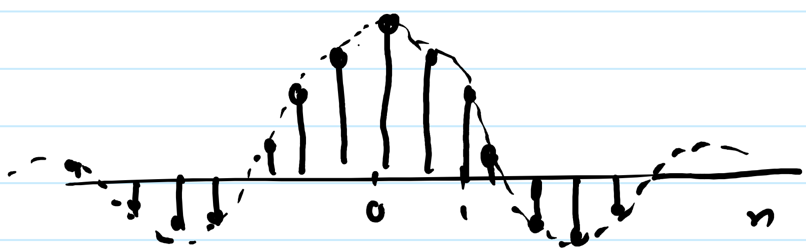
Consider ideal low pass filter

$$H(e^{j\omega}) = \begin{cases} 1 ; & |\omega| \le \omega_c \\ 0 ; & \omega_c < |\omega| \le \pi \end{cases}$$



impulse response of ideal LPF

$$h(n) = \frac{\sin(\omega_c n)}{\pi n}$$



$h(n)$ is IIR system
(infinite duration impulse response)

$h(n)$ is not causal

→ Not realizable in practice

h

Let us focus on practical systems

Specifically, systems described by

LCCDE

$$y(n) = \underbrace{- \sum_{k=1}^{N} a_k \, y(n-k)}_{\text{feedback terms}} + \underbrace{\sum_{k=0}^{M} b_k \, x(n-k)}_{\text{input terms}}$$

Taking z-transforms

$$Y(z) = - \sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

Transfer function

$$H(z) = \frac{Y(z)}{X(z)}$$

(rational)

$$= \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

We can write

$$H(z) = \frac{b_0 \prod_{k=1}^{M} (1 - q_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}$$

$$\prod_{k=1}^{N} (1 - p_k z^{-1})$$

$$H(z) = b_0 \, z^{N-M} \frac{\prod_{k=1}^{M} (z - q_k)}{\prod_{k=1}^{N} (z - p_k)}$$

$q_1, q_2, \ldots q_M$ are zeros

$p_1, p_2, \ldots, p_N$ are poles

$z^{N-M} \Rightarrow$ trivial pole or zero at $z = 0$ (depending on sign of $N-M$)

We restrict our attention to <u>Causal</u> & <u>stable</u> systems

$\Rightarrow$ All poles $p_1, p_2, \ldots p_N$ should be <u>inside</u> unit circle

$\Rightarrow$ ROC of ~~z-[illegible]s~~ transfer function $H(z)$ should lie outside the largest pole (& thereby contain the unit circle)

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Since ROC includes unit circle,

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

$$= b_0 e^{j\omega(N-M)} \frac{\prod_{k=1}^{M} (e^{j\omega} - q_k)}{\prod_{k=1}^{N} (e^{j\omega} - p_k)}$$

Magnitude response

$$|H(e^{j\omega})| = |b_0| \underbrace{|e^{j\omega(N-M)}|}_{=1} \frac{\prod_{k=1}^{M} |e^{j\omega} - q_k|}{\prod_{k=1}^{N} |e^{j\omega} - p_k|}$$

$$|H(e^{j\omega})| = |b_0| \frac{\prod_{k=1}^{M} |e^{j\omega} - q_k|}{\prod_{k=1}^{N} |e^{j\omega} - p_k|}$$

$$-\pi \leq \omega \leq \pi$$

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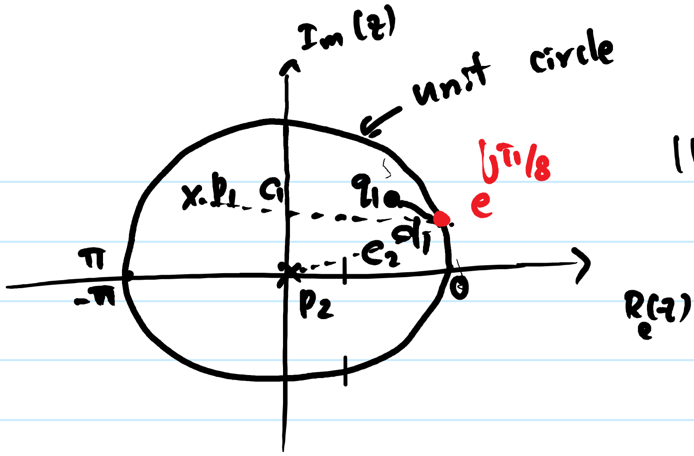

$$|H(e^{j\pi/8})| = \frac{d_1 |b_0|}{c_1}$$

Note $c_2 = 1$

Say $\omega = \pi/8$

$$|H(e^{j\pi/8})| = |b_0| \frac{\prod_{k=1}^{M} |e^{j\pi/8} - q_k|}{\prod_{k=1}^{N} |e^{j\pi/8} - p_k|}$$

Geometrically, $|e^{j\omega} - q_k|$ is distance between the point $e^{j\omega}$ on unit circle with the zero q_k

Numerator is the product of distances between $e^{j\omega}$ and the zeros $q_1, q_2, \ldots, q_M$

Denominator is product of distances between $e^{j\omega}$ and poles $p_1, \dots, p_N$

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Remarks

- Pole or zero at origin does not affect the magnitude response

$$|e^{j\omega} - 0| = 1$$

- Suppose there is zero q_k on unit circle;

$$q_k = e^{j\omega_0}$$

for some ω_0

Then $H(e^{j\omega_0}) = 0$

- In general, values of $\omega \in [-\pi, \pi]$ for which $e^{j\omega}$ is close to a zero of transfer function $|H(e^{j\omega})|$ will be small

- ~~Ex~~ On the other hand, $e^{j\omega}$ close to a pole will have large gain.

- It is convenient to plot the magnitude response in log scale (dB)

$$\text{Gain in dB} = 20 \log |H(e^{j\omega})|$$

$$= 20 \log |b_0| + \sum_{k=1}^{M} 20 \log |e^{j\omega} - q_k| - \sum_{k=1}^{N} 20 \log |e^{j\omega} - p_k|$$

(q_k are zeros
p_k are poles)

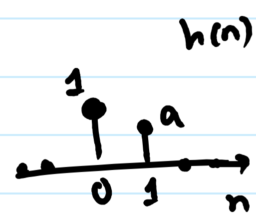
First order System: Single Complex zero

$$H(z) = 1 - a z^{-1}$$

$$a = r e^{j\theta}$$

$r \Rightarrow$ mag. of zero

$\theta \rightarrow$ angle of zero



FIR

$$Y(z) = H(z) X(z)$$
$$= (1 - a z^{-1}) X(z) \; ; \quad y(n) = x(n) - a x(n-1)$$
$$h(n) = \delta(n) - a \delta(n-1)$$

$$H(e^{j\omega}) = 1 - a e^{-j\omega}$$
$$= 1 - r e^{j\theta} e^{-j\omega}$$

Consider

$$|H(e^{j\omega})|^2 = H(e^{j\omega}) \cdot H^*(e^{j\omega})$$
$$= (1 - r e^{j\theta} e^{-j\omega})(1 - r e^{-j\theta} e^{j\omega})$$
$$= 1 + r^2 - 2r \cos(\omega - \theta)$$

$$20 \log |H(e^{j\omega})|$$
$$= 10 \log (1 + r^2 - 2r \cos(\omega - \theta))$$

Min of $|H(e^{j\omega})|$ occurs at $\omega = \theta$

$$|H_{min}|^2 = 1 + r^2 - 2r$$
$$= (1 - r)^2$$

Max of $|H(e^{j\omega})|$ occurs at $\omega = \theta + \pi$

$$|H_{max}|^2 = 1 + r^2 + 2r$$
$$= (1 + r)^2$$

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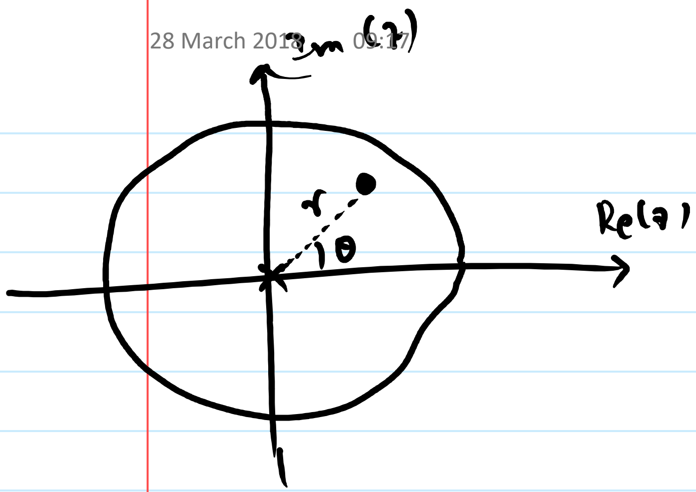
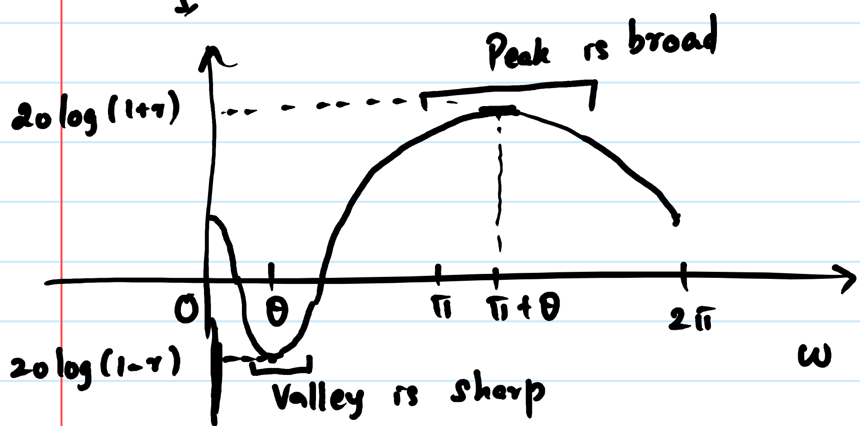


$$20\log|H(e^{j\omega})| = 10\log(1+r^2-2r\cos(\omega-\theta))$$



Single complex Pole

$H(z) = \dfrac{1}{1-az^{-1}}$;

causal, stable

$|a| < 1$

$h(n) = a^n u(n)$

$Y(z)(1-az^{-1}) = X(z)$

$y(n) = x(n) + \underbrace{a\,y(n-1)}_{\text{feedback}}$

IIR System
Causal } implementable

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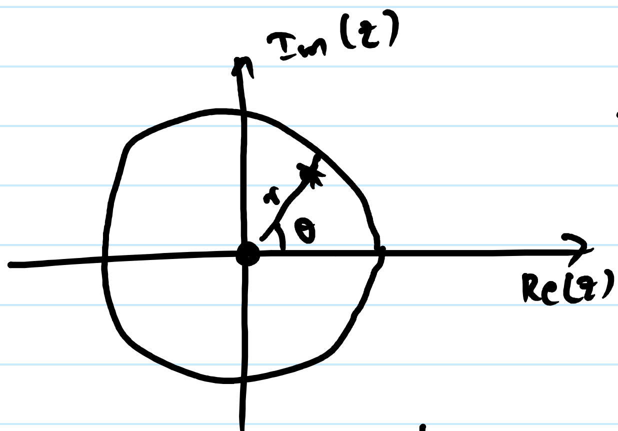
$$a = r e^{j\theta}$$

$$H(e^{j\omega}) = \frac{1}{1 - re^{j\theta} e^{-j\omega}}$$

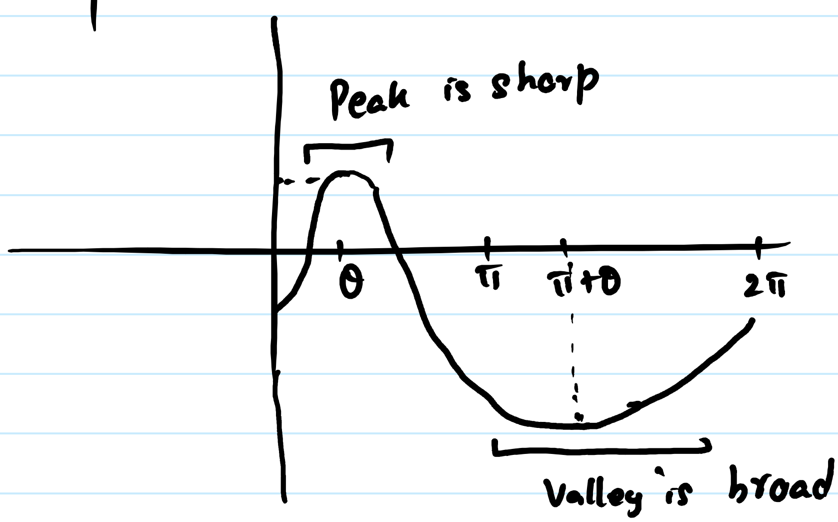
$$20 \log |H(e^{j\omega})| = -20 \log |1 - r e^{j\theta} e^{-j\omega}|$$

$$= -10 \log (1 + r^2 - 2r \cos(\omega - \theta))$$

Same as the single zero except for negative sign



$r < 1$



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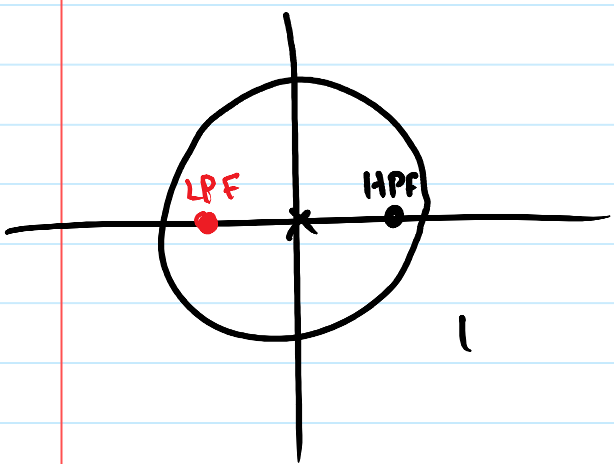
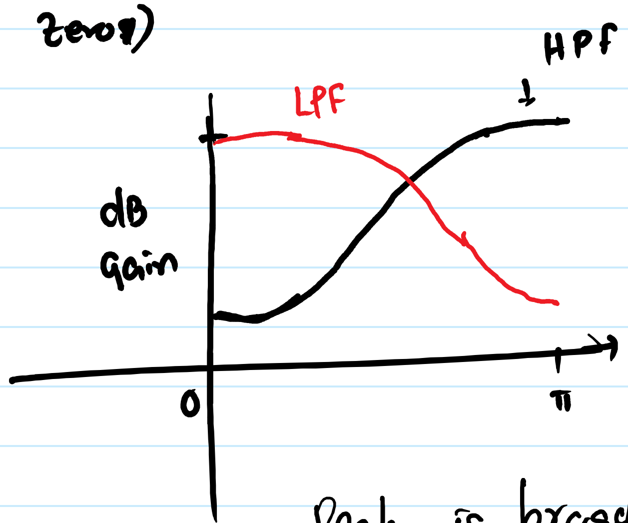
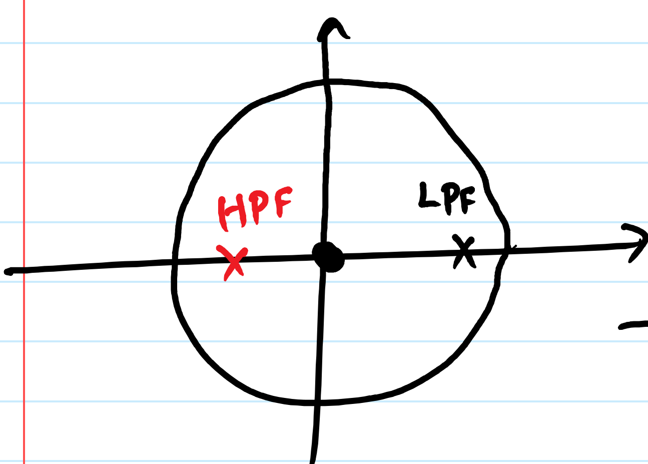
Low Pass + High Pass filters
(using zeros)

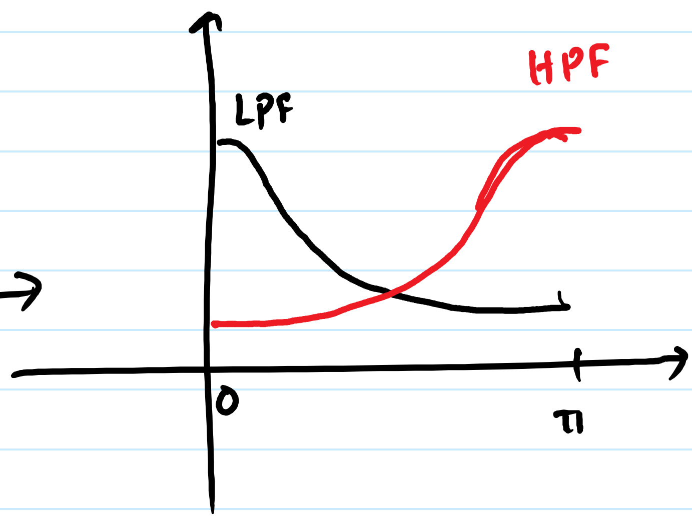




Peak is broad
Valley is sharp

LPF & HPF using pole





Valley is broad
Peak is sharp

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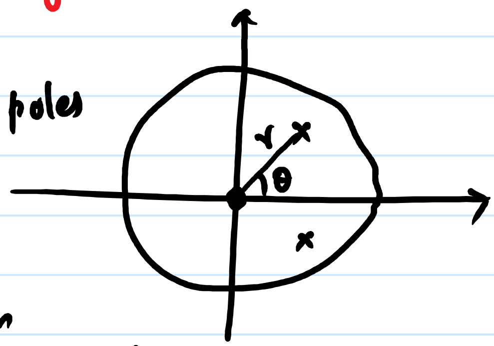
Second Order System

Two Conjugate poles

$re^{j\theta}, re^{-j\theta}$



This system is called a RESONATOR

$$H(z) = \frac{1}{(1 - az^{-1})(1 - a^{*}z^{-1})}$$

where $a = re^{j\theta}$

Can show that

$$h(n) = \frac{r^{n}\sin(n+1)\theta}{\sin\theta} u(n)$$

$$|H(e^{j\omega})| = \frac{1}{|1 - ae^{-j\omega}|} \cdot \frac{1}{|1 - a^{*}e^{-j\omega}|}$$

pole @ $re^{j\theta}$

$$20\log|H(e^{j\omega})|$$

$$= -20\log|1 - re^{j\theta}e^{-j\omega}|$$

$$- 20\log|\underbrace{1 - re^{-j\theta}}_{\text{Pole @ } re^{-j\theta}} e^{-j\omega}|$$

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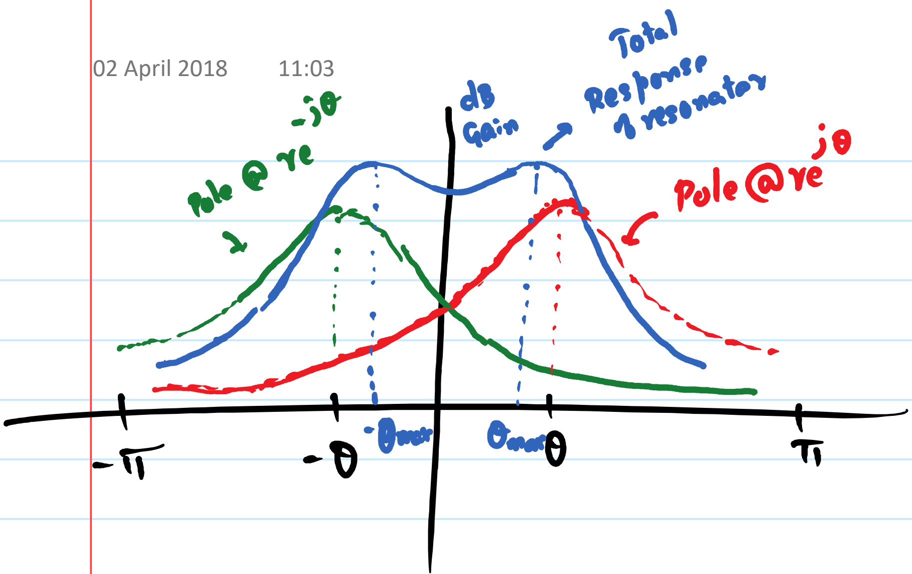


We can show that

$$\theta_{max} = \cos^{-1}\left(\frac{1+r^2}{2r} \cos\theta \right)$$

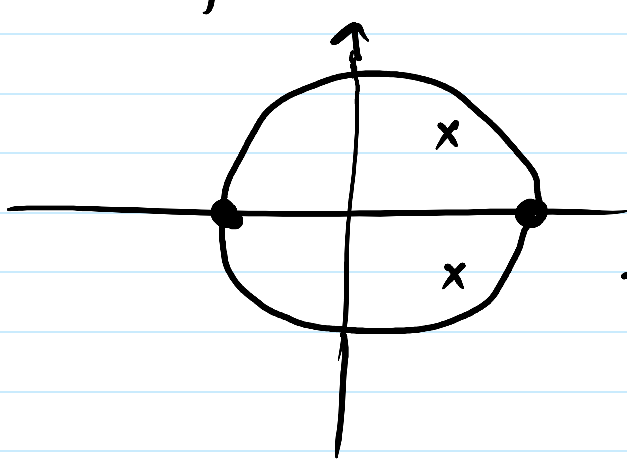
As $r \to 1$, we have

$\theta_{max} \to \theta$

If $\theta = \pi/2$ then $\theta_{max} = \pi/2$

for any r

Band Pass filter using resonator



Poles at

$re^{j\theta}, re^{-j\theta}$

Zeros at

$1, -1$

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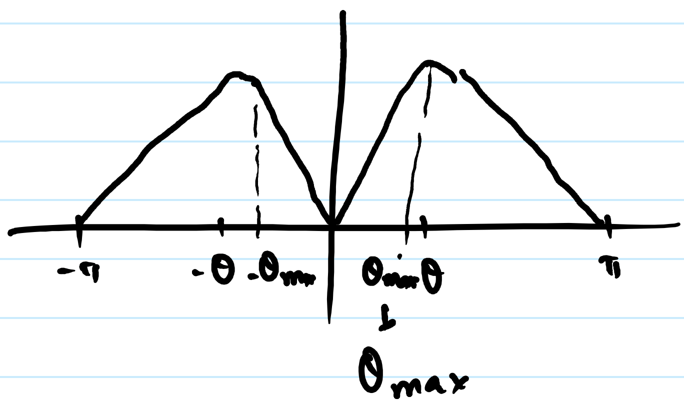
$$H(z) = \frac{(1-z^{-1})(1+z^{-1})}{(1-az^{-1})(1-a^{*}z^{-1})}$$

$a = re^{j\theta}$



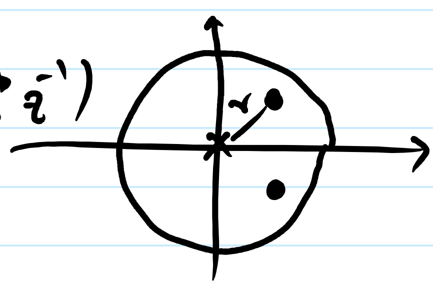
$$\theta_{max} = \cos^{-1}\left(\frac{2r}{1+r^2}\cos\theta\right)$$

<u>Notch filter</u> (Second order)

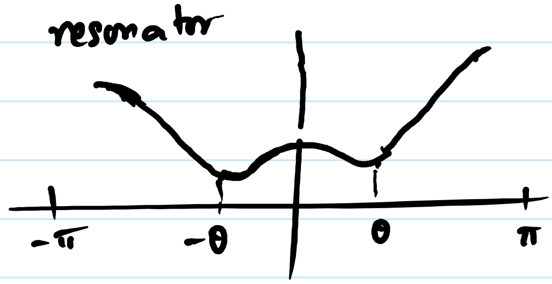
Zeros at $re^{j\theta}$, $re^{-j\theta}$

$H(z) = (1-az^{-1})(1-a^{*}z^{-1})$

$a = re^{j\theta}$



Magnitude response (dB) is negative of resonator



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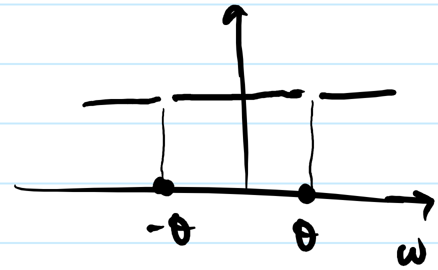
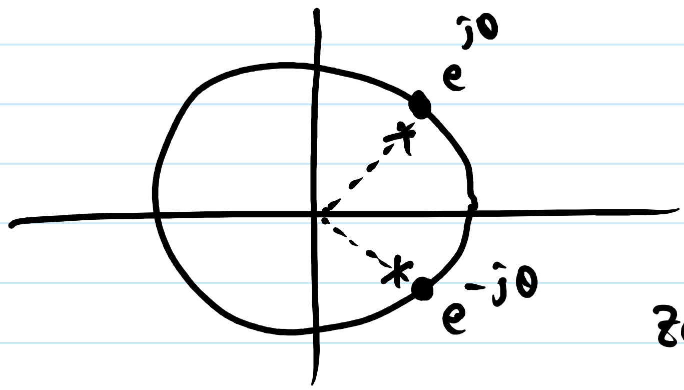
Improved Notch filter

Ideal Notch filter

$$|H(e^{j\omega})| = \begin{cases} 0 & \text{if } \omega = \pm\theta \\ 1 & \text{if } |\omega| \neq \theta \end{cases}$$





Zeros at $e^{j\theta}, e^{-j\theta}$

Poles at $re^{j\theta}, re^{-j\theta}$

$$H(z) = \frac{1 - 2\cos\theta z^{-1} + z^{-2}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

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Moving Average Filter (FIR)

$$h(n) = \begin{cases} 1/N , & 0 \le n \le N-1 \\ 0 ; & \text{else} \end{cases}$$

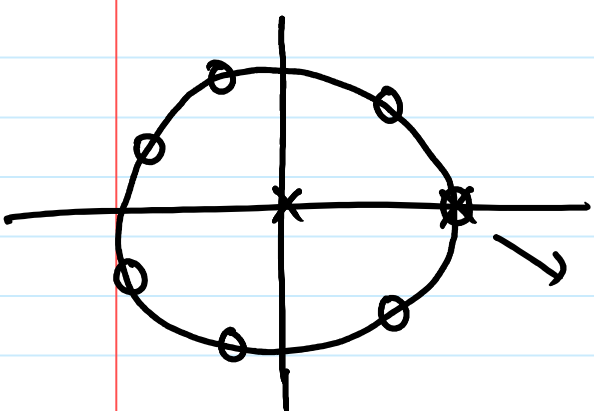
$$H(z) = \frac{1}{N}\left(1 + z^{-1} + \dots + z^{-(N-1)}\right)$$

$$= \frac{1}{N} \frac{1 - z^{-N}}{1 - z^{-1}}$$

No. of zeros is governed by N

$N = 7$

LPF



Pole zero cancellation at $z = 1$