Note Title 24-08-2015

Linear Time-Invariant Systems

(LTI)

An LTI system is both linear and time-invariant

Consider Discrete-time Case

$$x(n) \longrightarrow \boxed{\text{DT system LTI}} \longrightarrow y(n)$$

If $x_1[n] \longrightarrow y_1[n]$, $x_2[n] \longrightarrow y_2[n]$

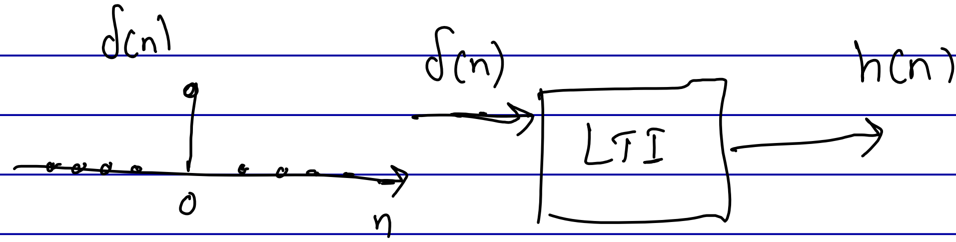
then (linearity) $a\,x_1[n] + x_2[n] \longrightarrow a\,y_1[n] + y_2[n]$

(time invariance) $x_1(n-N) \longrightarrow y_1(n-N)$

for any a, N, $x_1(n)$, $x_2(n)$

We will characterize (complete & equivalent) the LTI systems using impulse response

Impulse response of an LTI system (denoted by $h[n]$) is defined as the output of the system when the input is unit-impulse function $\delta(n)$.



CLAIM: By knowing impulse response $h(n)$ of LTI system, we can compute the system output for any input signal.

We will show this using two ideas.

① Any signal $x(n)$ can be written as a sum of (scaled) time-shifted impulse function.

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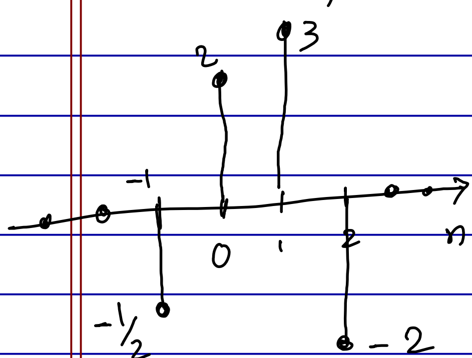
(2) For an LTI system, Sum of time shifted inputs produces Sum of time-shifted outputs.

x ———— x

(1) Writing $x(n)$ in terms of delta function $\delta(n)$.



$$x(n) = \begin{cases} -1/2 & ; n = -1 \\ 2 & ; n = 0 \\ 3 & ; n = 1 \\ -2 & ; n = 2 \\ 0 & \text{otherwise} \end{cases}$$

clearly $x(n) = (-1/2)\,\delta(n+1) + 2\delta(n) + 3\,\delta(n-1) + (-2)\,\delta(n-2)$

$= x(-1)\,\delta(n-(-1)) + x(0)\,\delta(n-0) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2)$

general term $x(k)\,\delta(n-k)$

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Generalizing this idea, for any signal $x(n)$ we have

$$\boxed{x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)}$$

complete signal — $x(n)$

sample value at time k — $x(k)$

$\delta(n)$ shifted by k time-units — $\delta(n-k)$

Computing output of LTI system with impulse response $h(n)$ for an arbitrary input $x(n)$

For input $\delta(n)$ output is $h(n)$

$$\delta(n) \longrightarrow h(n)$$

Time shifted impulse $\delta(n-k) \xrightarrow[\text{time-invariance}]{} h(n-k)$

$x(k)\,\delta(n-k) \xrightarrow[\text{homogeneity of LINEAR systems}]{} x(k)\,h(n-k)$

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$$\sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \longrightarrow \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$x(n) \qquad\qquad\qquad y(n)$$

The output $y(n)$ corresponding to input $x(n)$ (for LTI system with impulse response $h(n)$) is given by

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)}$$

This called convolution sum

In shorthand, it is denoted by $*$

$$y(n) = x(n) * h(n)$$

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Given $x(n)$ & $h(n)$ how to
input impulse response

compute $y(n) = x(n) * h(n)$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Procedure:

1. Fix the value of n

2. Think $x(k)$ & $h(n-k)$ as DT time signals indexed by k

3. Flip (time reverse) the sequence (impulse response) $h(k)$ to get $h(-k)$

4) Time-shift the flipped sequence to get $h(n-k)$

5) Compute the product of signals $x(k)$ and $h(n-k)$ and get $x(k)\,h(n-k)$

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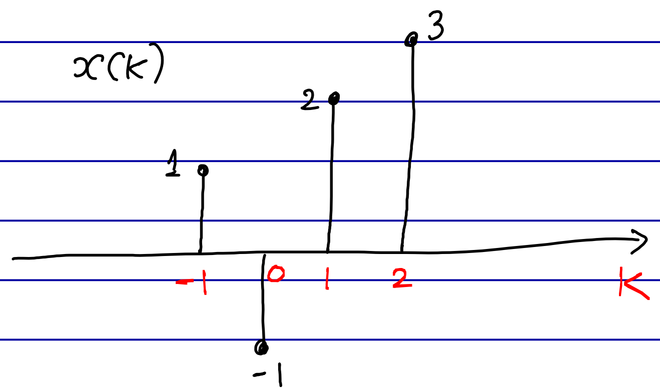
6. Sum the values of signal $x(k)\,h(n-k)$ and get $y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$

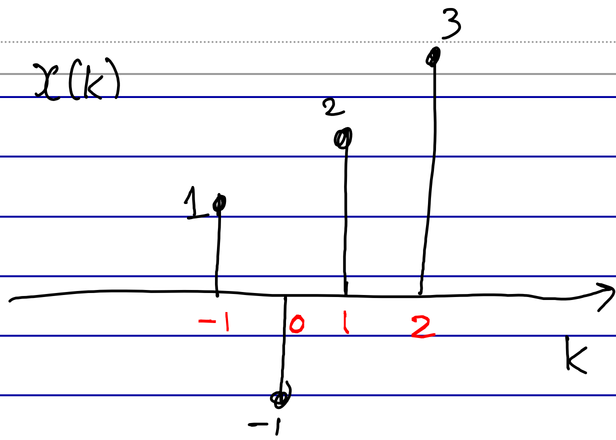
7. Repeat this process by changing n and get the entire Output signal $y(n)$

Example:

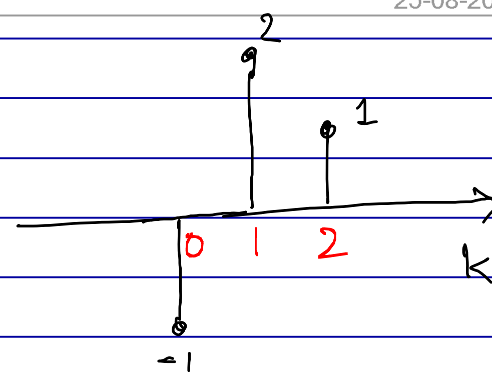
$$x(n) = \begin{cases} 1 & \text{for } n=-1 \\ -1 & \text{for } n=0 \\ 2 & \text{for } n=1 \\ 3 & \text{for } n=2 \\ 0 & \text{otherwise} \end{cases} \qquad h(n) = \begin{cases} -1 & \text{for } n=0 \\ 2 & \text{for } n=1 \\ 1 & \text{for } n=2 \\ 0 & \text{otherwise} \end{cases}$$



$x(k)$

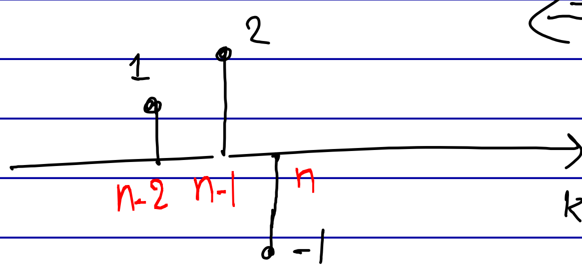


$h(k)$

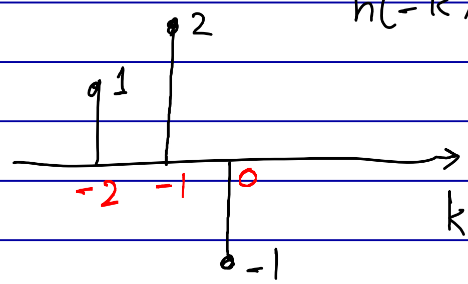


Flip

$h(n-k)$ Shift



$h(-k)$

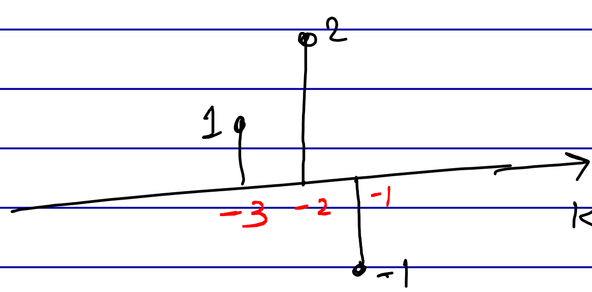


If $n \leq -2$, the product $x(k)\,h(n-k)$ is always zero

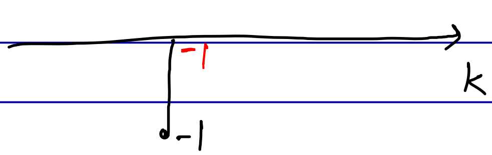
$y(n) = 0$ for $n \leq -2$, and $n \geq 5$

Consider $n = -1$;

$h(-1-k)$



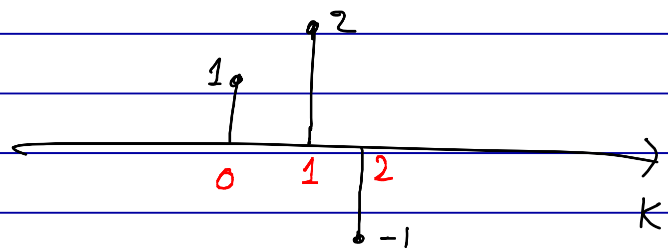
$x(k)\,h(-1-k)$



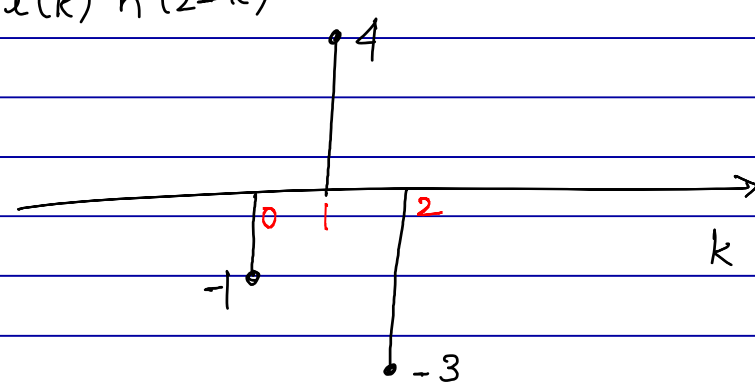
$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)\,h(-1-k) = -1$$

Consider $\underline{\underline{n=2}}$

$h(2-k)$



$x(k)\,h(2-k)$

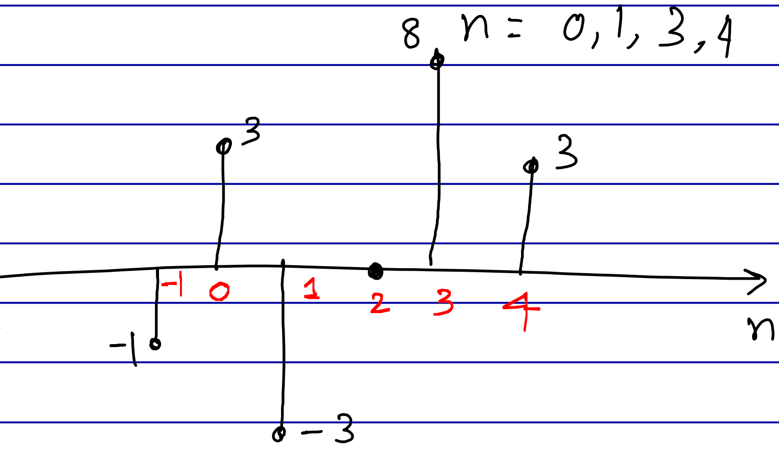


$$y(2) = \sum_{k=-\infty}^{\infty} x(k)\,h(2-k) = 0$$

Similarly we can find $y(n)$ for

$n = 0, 1, 3, 4$

$\underline{\underline{\text{Verify}}}$ $y(n)$



$y(n) = 0$ for

$n \le -2$ and

$n \geq 5$

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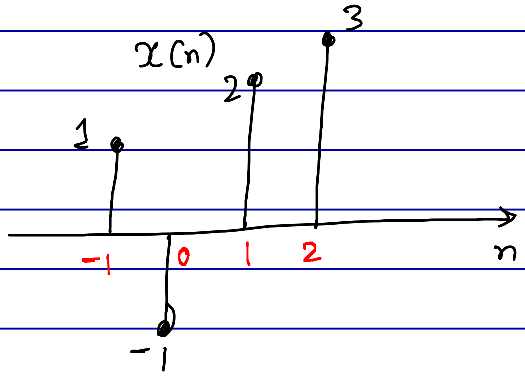
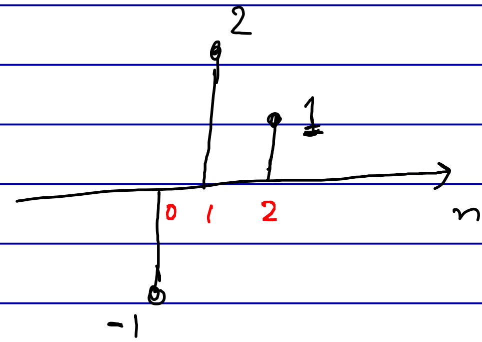
An alternative approach to obtain output using convolution.

$x(n)$

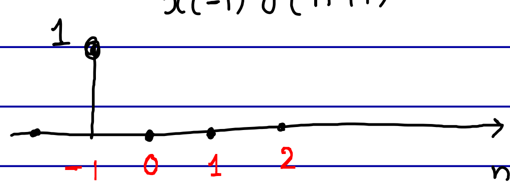


$h(n)$



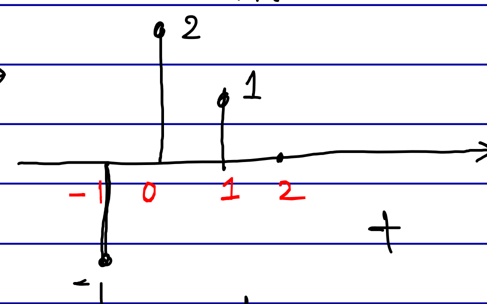
$$\delta(n) \longrightarrow h(n)$$

$x(-1)\,\delta(n+1)$



$\longrightarrow$

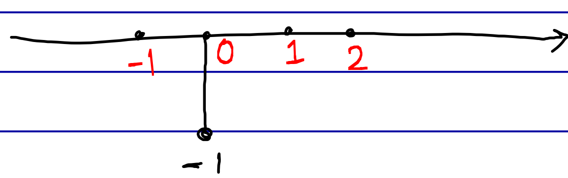
$1 \cdot h(n+1)$



$+$

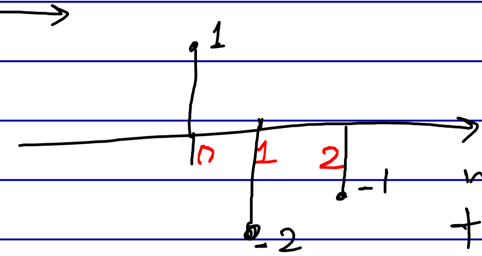
$+$

$x(0)\,\delta(n)$



$\longrightarrow$

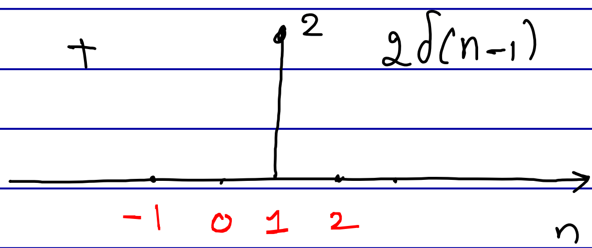
$-\,h(n)$



$+$

$+$

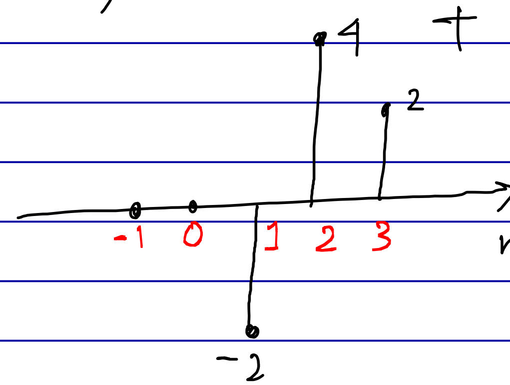
$2\,\delta(n-1)$



$\longrightarrow$

$2\,h(n-1)$

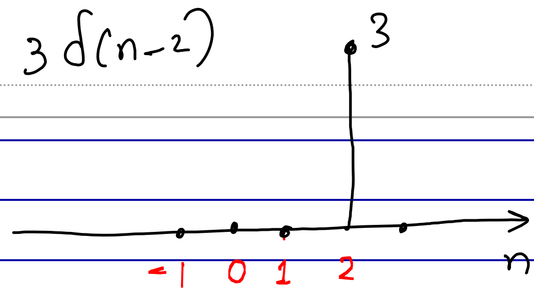
$+$



$+$

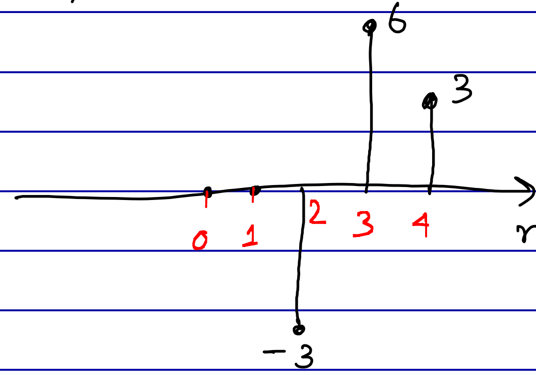
$+$

$3\delta(n-2)$



$\longrightarrow$ $3h(n-2)$

$+$



$= x(n)$

$= y(n)$

$x(n) \longrightarrow y(n)$

Verify that you get the same output as before.

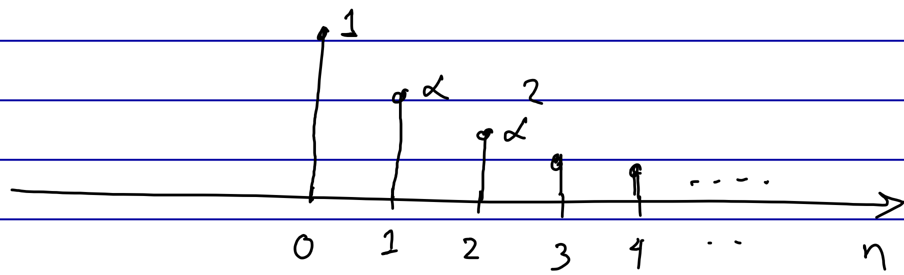
Suppose input $x(n)$ has N_1 Samples

impulse response $h(n)$ has N_2 Samples

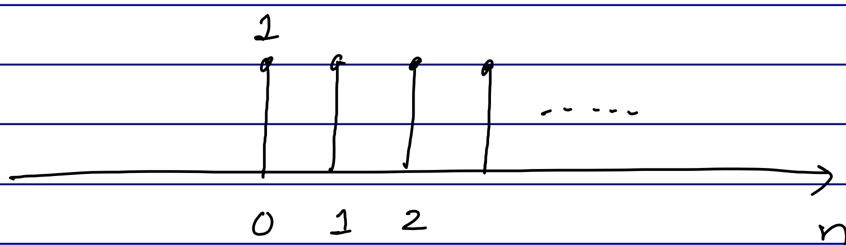
Output $y(n) = x(n) * h(n)$ will have N_1+N_2-1 samples

Example with infinite-length $x(n)$ & $h(n)$

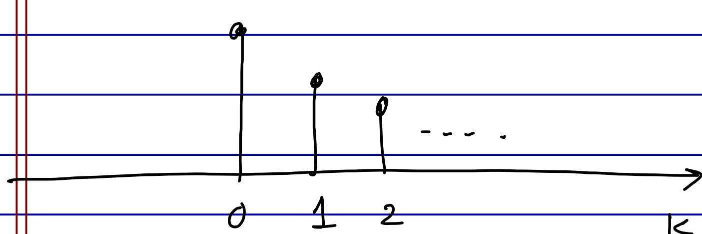
$$x(n) = \alpha^n u(n) \quad ; \quad |\alpha| < 1$$

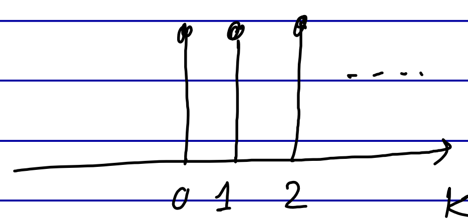


$$h(n) = u(n)$$

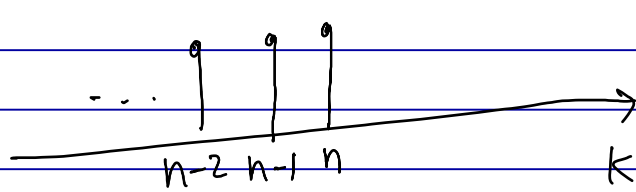


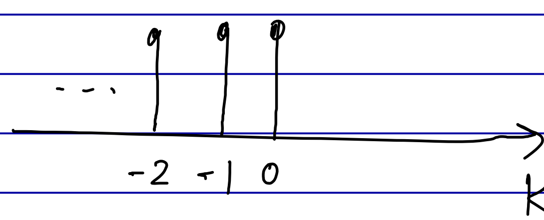
$x(k)$ $\alpha^k u(k)$ $h(k) = u(k)$





$h(n-k)$ $h(-k)$





Note Title 26-08-2015

If $n < 0$, then

$x(k)\, h(n-k)$ will always be zero

$y(n) = 0$ for $n < 0$

For $n \geq 0$;

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=0}^{n} x(k)\, h(n-k)$$

$x(k)$ is zero for $k < 0$

$h(n-k)$ is zero for $k > n$

$$= \sum_{k=0}^{n} \alpha^{k} \cdot 1$$

$$= \frac{1-\alpha^{n+1}}{1-\alpha}$$

$$y(n) = \frac{1-\alpha^{n+1}}{1-\alpha}\, u(n)$$

Note Title 31-08-2015

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Commutativity Property of Convolution

Let $h(n)$ be impulse response of an LTI System

Let $x(n)$ be an input to the System

Output $y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$

$x(n) * h(n)$

Change of Variables

$l = n - k$

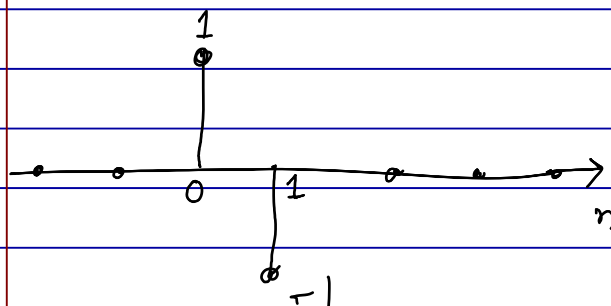
$$= \sum_{l=\infty}^{-\infty} x(n-l)\, h(l)$$

$$y(n) = \sum_{l=-\infty}^{\infty} h(l)\, x(n-l)$$

$$\boxed{x(n) * h(n) = h(n) * x(n)}$$

Example: LTI system

with
$$h(n) = \begin{cases} 1 & \text{for } n=0 \\ -1 & \text{for } n=1 \\ 0 & \text{otherwise} \end{cases}$$



For input $x(n)$, let us find output $y(n)$

$$y(n) = x(n) * h(n)$$

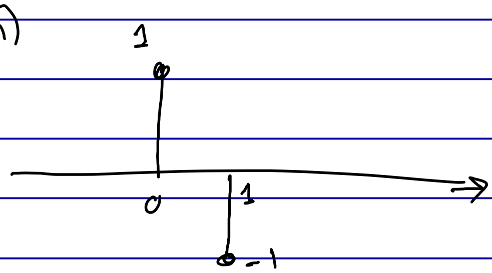
$$= h(n) * x(n)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$= h(0)\, x(n-0) + h(1)\, x(n-1)$$

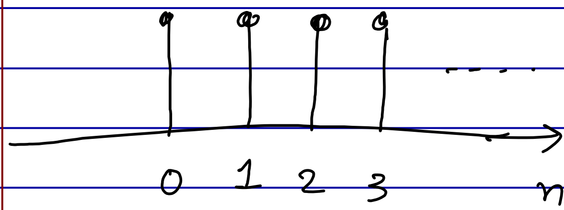
$$y(n) = x(n) - x(n-1)$$

$h(n)$



(Equivalent) $\Longleftrightarrow$ $y(n) = x(n) - x(n-1)$

$h(n) = u(n)$



$\Longleftrightarrow y(n) = x(n) + x(n-1) + x(n-2) + \cdots$

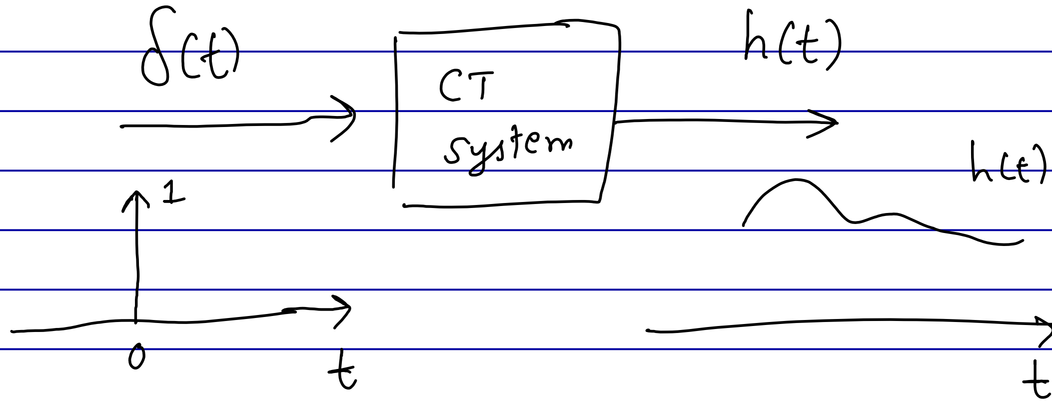
$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

(Accumulator)

x ——————————— x

Continuous - time Systems

Impulse response of a CT-system is the output of the system when the input is unit impulse function



Note Title 31-08-2015

For an LTI system with impulse response $h(t)$, the output $y(t)$ for input $x(t)$ is given by

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau}$$

$\Downarrow$

Convolution integral.

$$y(t) = x(t) * h(t)$$

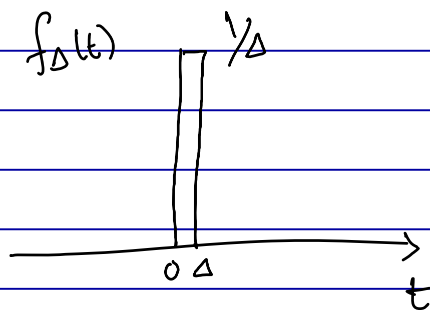
$$= h(t) * x(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Derivation will be <u>nearly</u> same as DT case

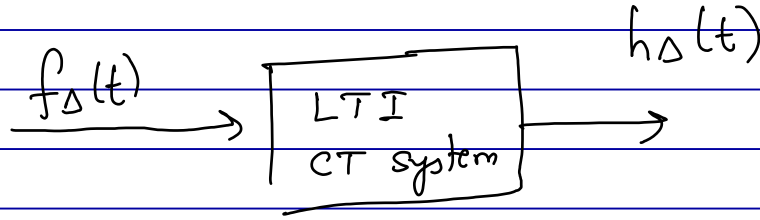
Note Title 31-08-2015

Consider $f_\Delta(t)$



$$f_\Delta(t) = \begin{cases} 1/\Delta & ; 0 \le t \le \Delta \\ 0 & ; \text{else.} \end{cases}$$

$$\lim_{\Delta \to 0} f_\Delta(t) = \delta(t)$$



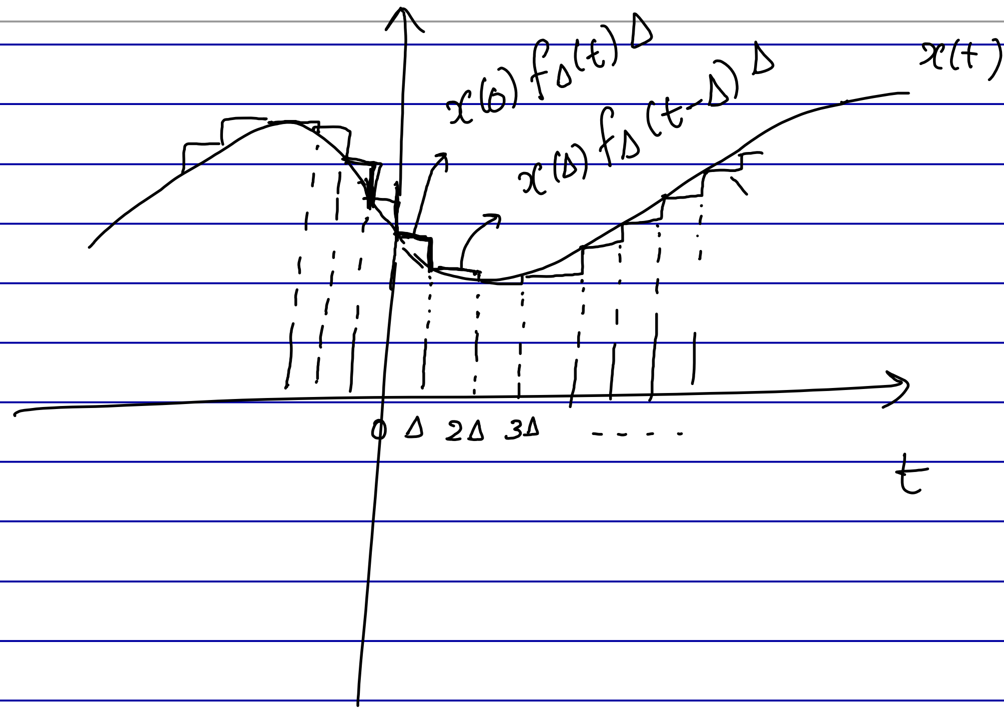
Let $h_\Delta(t)$ be output of system with input $f_\Delta(t)$

$$\lim_{\Delta \to 0} f_\Delta(t) = \delta(t) \quad \text{unit impulse}$$

$$\lim_{\Delta \to 0} h_\Delta(t) = h(t) \quad \text{impulse response}$$

Let $x(t)$ be an arbitrary input

We will approximate $x(t)$ using shifted copies of $f_\Delta(t)$



$\hat{x}(t)$ = staircase approximation of $x(t)$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta) f_\Delta(t-k\Delta)\Delta$$

$\lim_{\Delta\to 0}$ $\lim_{\Delta=0}$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

(Sifting property)

$$= \frac{1}{a}\left(1 - e^{-at}\right)$$

Note Title 31-08-2015

For the given LTI system

$$\text{input } f_\Delta(t) \longrightarrow h_\Delta(t) \text{ output}$$

$$\text{Shifting } f_\Delta(t-k\Delta) \longrightarrow h_\Delta(t-k\Delta)$$

(time invariance)

$$x(k\Delta)\, f_\Delta(t-k\Delta)\,\Delta \longrightarrow x(k\Delta)\, h_\Delta(t-k\Delta)\,\Delta$$

(homogeneity)

$$\sum_{k=-\infty}^{\infty} x(k\Delta) f_\Delta(t-k\Delta)\,\Delta \longrightarrow \sum_{k=-\infty}^{\infty} x(k\Delta) h_\Delta(t-k\Delta)\,\Delta$$

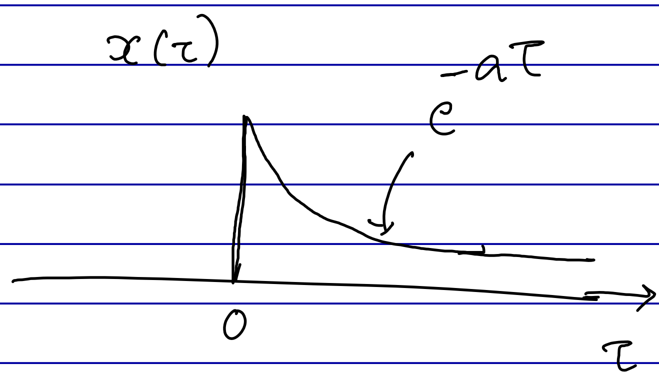
$$\Big\downarrow \lim_{\Delta\to 0} \qquad\qquad \Big\downarrow \lim_{\Delta\to 0}$$

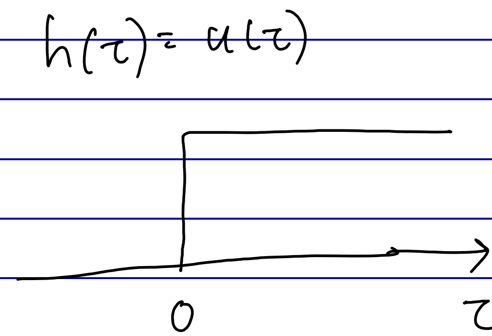
$$x(t) \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

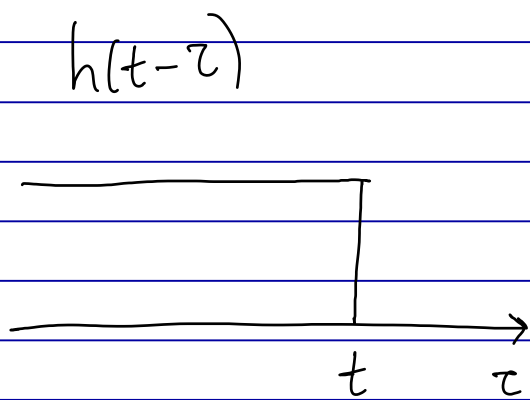
$$x \longrightarrow x$$

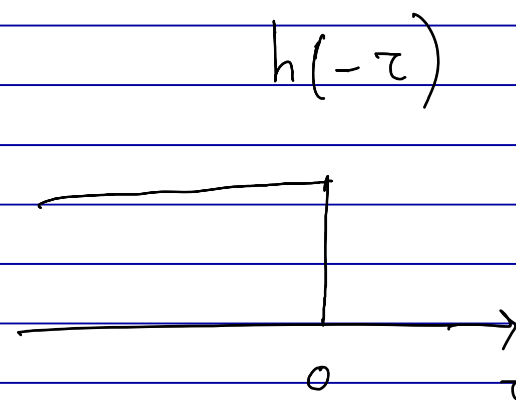
Example: $x(t) = e^{-at} u(t) \qquad a > 0$

$h(t) = u(t)$









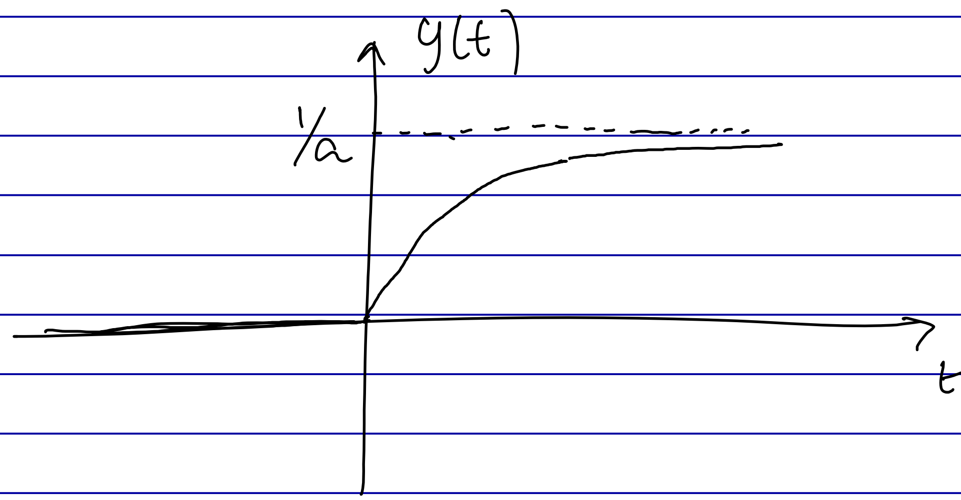
For $t < 0$, $x(\tau)\, h(t-\tau)$ will be zero

For $t > 0$; $x(\tau)\, h(t-\tau)$ will be non zero for $\tau = 0$ to t

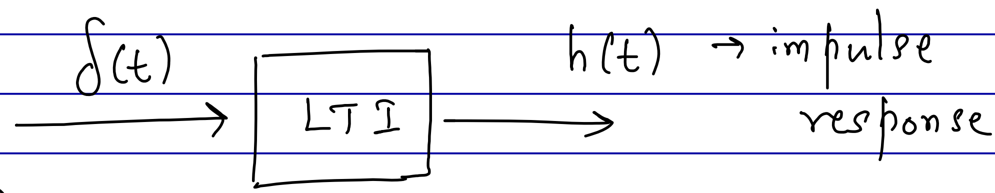
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

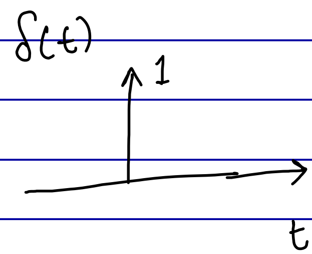
$$= \int_0^t e^{-a\tau} \cdot 1 \cdot d\tau$$

$$y(t) = \frac{1}{a}\left(1 - e^{-at}\right) u(t)$$



1/9 Practical interpretation of impulse response





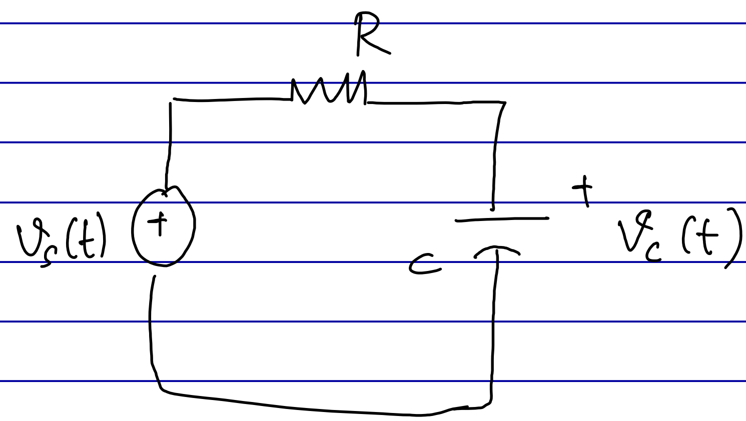
$\delta(t)$ is an idealized function.

$\delta(t)$ can not be generated in a real system

Interpreting impulse response of a real system

Illustrative Example:

RC circuit



The input is $V_s(t)$ denote by $x(t)$

The output is $V_c(t)$ denote by $y(t)$

Assume zero initial conditions
(Initial capacitor voltage is zero)

$$RC\frac{dv_c(t)}{dt} + V_c(t) = V_s(t)$$

$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

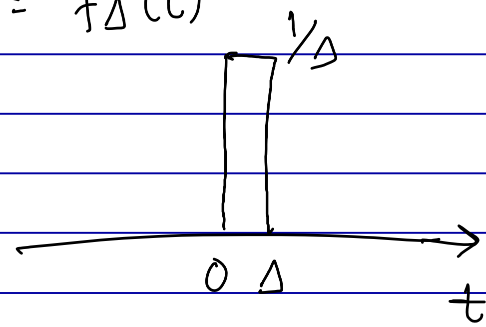
input-output relationship

Verify that this is LTI

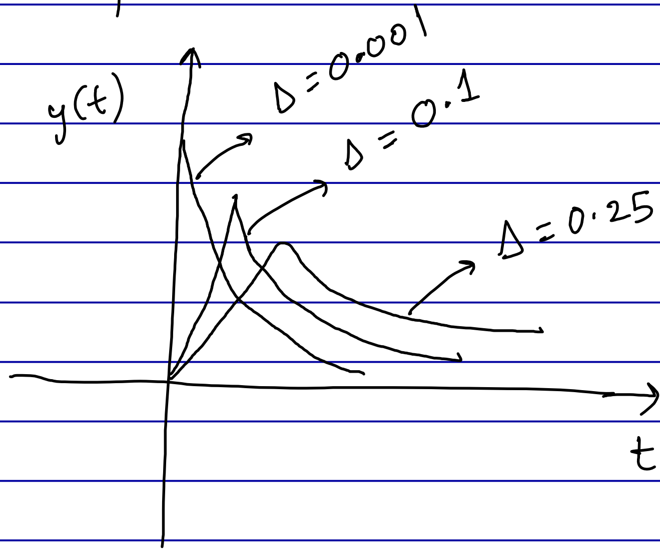
For convenience Assume $RC = 1$

Let us find impulse response of system.

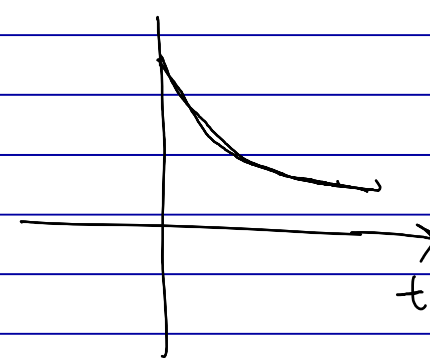
Suppose $x(t) = f_\Delta(t)$



Output $y(t)$



$h(t) = e^{-t}\,u(t)$



As $\Delta \to 0$, $y(t) \longrightarrow e^{-t}\,u(t)$

$\Downarrow$

impulse response

In this example, for $\Delta = 0.001$ the output $y(t)$ is close to $h(t)$

for some other values of R & C we may need to choose different Δ.

Nevertheless,

for any system we may find a suitable Δ (small enough) so that output $y(t)$ is very close to impulse response $h(t)$.

$\delta(t)$ is an idealized signal that is short-enough for all systems

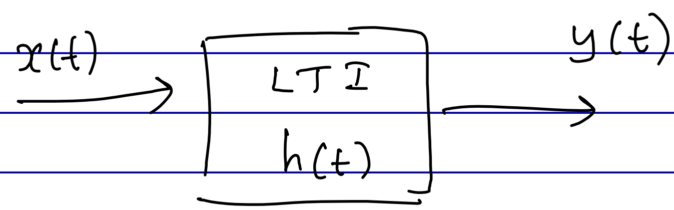
Properties of Convolution

① Commutative property

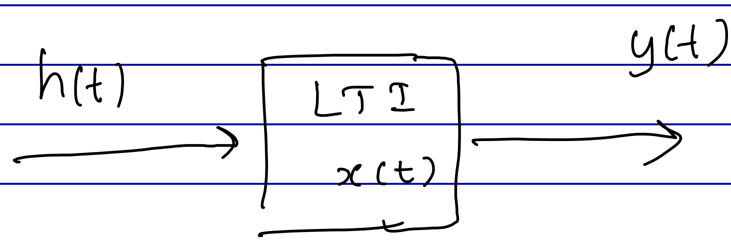
$$x(n) * h(n) = h(n) * x(n)$$

$$x(t) * h(t) = h(t) * x(t)$$

Block diagram



$\Downarrow$ will give same output

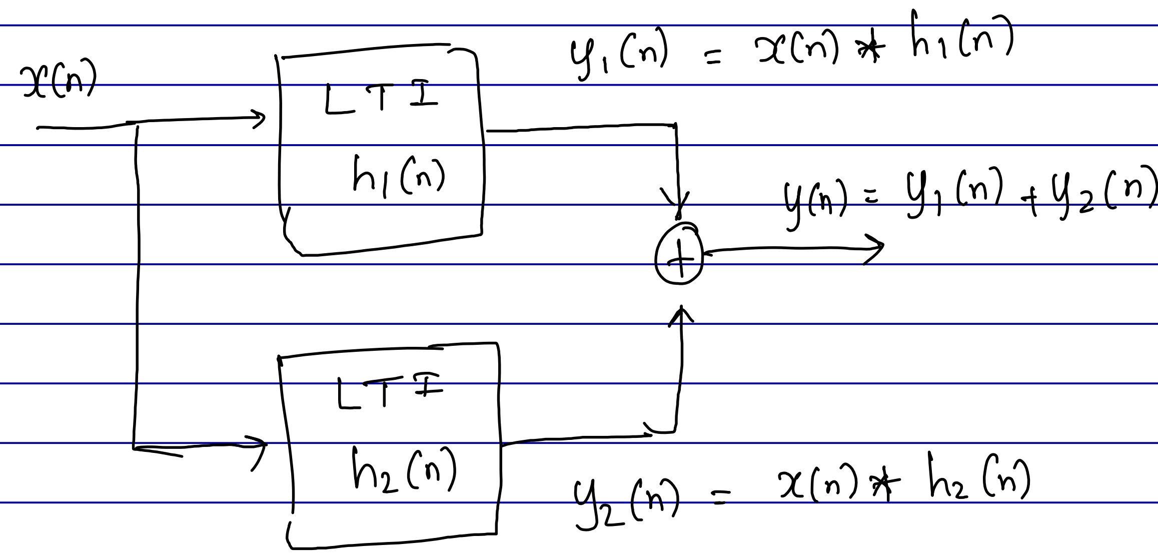


② Distributivity

$$x(n) * h_1(n) + x(n) * h_2(n)$$

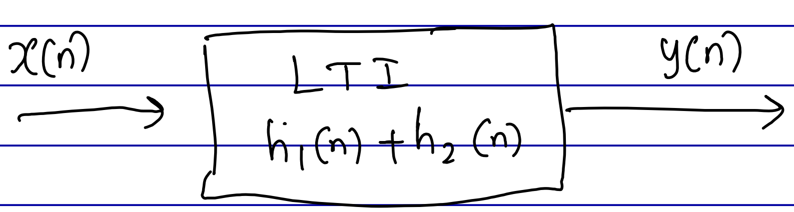
$$= x(n) * \left(h_1(n) + h_2(n) \right)$$

LHS



equivalent

RHS



Proof is simple. Try yourself

In CT case: $x(t) * \left(h_1(t) + h_2(t) \right)$

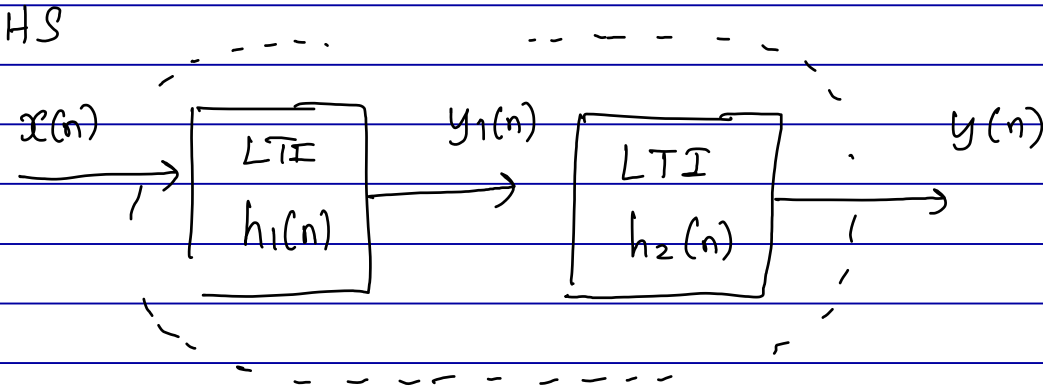
$$= x(t) * h_1(t) + x(t) * h_2(t)$$

③ Associativity

$$\big(x(n) * h_1(n)\big) * h_2(n)$$

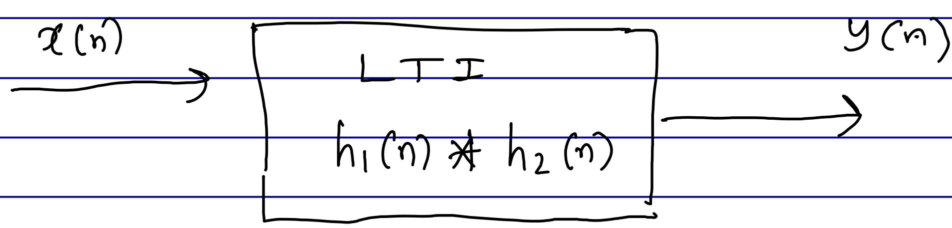
$$= x(n) * \big(h_1(n) * h_2(n)\big)$$

LHS



$\Downarrow$ equivalent to

RHS



When we cascade two LTI systems, the effective system is also LTI. The impulse response of this effective system is convolution of individual impulse responses

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Associativity of Convolution

$$\left(x(n) * h_1(n) \right) * h_2(n)$$

$$= x(n) * \left(h_1(n) * h_2(n) \right)$$

Proof:

Define $y_1(n) = x(n) * h_1(n)$

$h_3(n) = h_1(n) * h_2(n)$

Now, $y_1(n) = \sum_{k=-\infty}^{\infty} x(k)\, h_1(n-k)$ ($)

$h_3(n) = \sum_{l=-\infty}^{\infty} h_2(l)\, h_1(n-l)$ ($$)

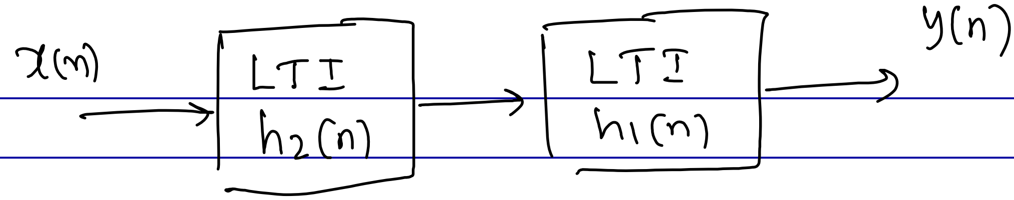
LHS $\left(x(n) * h_1(n) \right) * h_2(n)$

$= y_1(n) * h_2(n)$

$$= \sum_{l=-\infty}^{\infty} h_2(l)\, \underbrace{y_1(n-l)}_{\displaystyle \shortparallel \atop \displaystyle \sum_{k=-\infty}^{\infty} x(k)\, h_1(n-l-k)}$$

replace n with n-l in ($)

$\Downarrow$ Equivalent



$$LHS = \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h_2(l)\, x(k)\, h_1(n-l-k)$$

$$= \sum_{k=-\infty}^{\infty} x(k) \underbrace{\sum_{l=-\infty}^{\infty} h_2(l)\, h_1(n-k-l)}_{h_3(n-k)}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h_3(n-k)$$

$$= x(n) * h_3(n)$$

$$\left(x(n) * h_1(n)\right) * h_2(n) = x(n) * \left(h_1(n) * h_2(n)\right)$$

Same result holds for CT systems

$$\left(x(t) * h_1(t)\right) * h_2(t) = x(t) * \left(h_1(t) * h_2(t)\right)$$

An important Consequence:

When we cascade many LTI Systems together, the effective system is also LTI and the <u>order</u> of cascading systems <u>does not</u> <u>matter</u>.

<u>Proof:</u>

$$x(n) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h_1(n)\end{matrix}} \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h_2(n)\end{matrix}} \longrightarrow y(n)$$

(Associativity) $\Big\Downarrow$ Equivalent

$$x(n) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h_1(n) * h_2(n)\end{matrix}} \longrightarrow y(n)$$

Commutativity $\Big\Downarrow$ Equivalent

$$x(n) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h_2(n) * h_1(n)\end{matrix}} \longrightarrow y(n)$$

$\Big\Downarrow$

Connection between LTI system properties and its impulse response.

Let $h(n)$ be impulse response of an LTI system

Memoryless, causality, invertibility, Stability of LTI Systems can be inferred from its impulse response

① Memoryless LTI System

Output $y(n)$ should depend only on input at time n (not on past or future input Values)

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

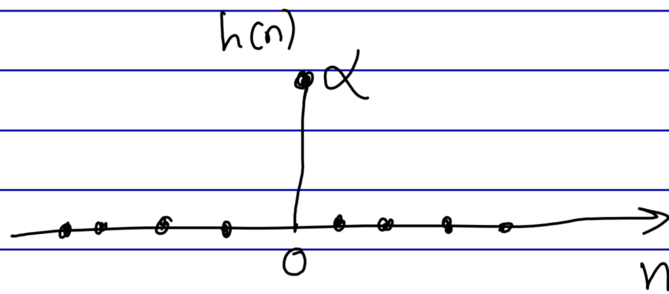
$$= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$y(n) = \cdots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + \cdots$$

If System is memoryless

$h(k) = 0$ for all non-zero values of k.

Let $h(0) = \alpha$



For this impulse response,

$$y(n) = h(0)x(n) = \alpha x(n)$$

Note $h(n) = \alpha\,\delta(n)$

Take $\alpha = 1$;

$$y(n) = x(n) * \delta(n) = x(n)$$

$$\Downarrow$$

$$\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) = x(n)$$

$$\Downarrow$$

Already seen this.

Equivalently in the CT case,

LTI system is memoryless

if its impulse response $h(t)$

is such that $h(t) = \alpha \, \delta(t)$

Again, $x(t) * \delta(t) = x(t)$

$$\int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau = x(t)$$

Sifting property of δ.

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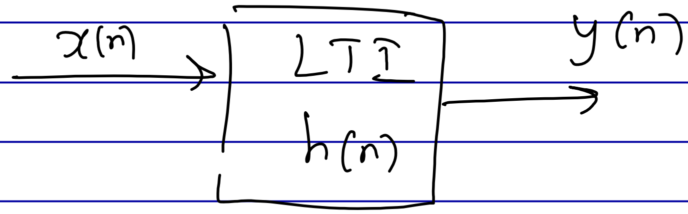
Causality of LTI Systems



$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$= \cdots + \underbrace{h(-1)x(n+1)}_{\text{future input values}} + \underbrace{h(0)\,x(n)}_{\text{present}} + \underbrace{h(1)\,x(n-1) + \cdots}_{\text{past}}$$

An LTI system is causal if its impulse response $h(n) = 0$ for all $n < 0$

Similarly in CT case, causal LTI system satisfies $h(t) = 0$ for all $t < 0$.

A signal $f(t)$ is called causal if $f(t) = 0$ for all $t < 0$

For causal LTI Systems

$$y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k)$$

Convolution for causal systems.

$l = n-k$.

$$y(n) = \sum_{l=-\infty}^{n} x(l)\, h(n-l)$$

Same thing in CT causal LTI systems

$$y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Stability of LTI System:

$$\xrightarrow{x(n)} \boxed{\begin{matrix}\text{LTI} \\ h(n)\end{matrix}} \xrightarrow{y(n)}$$

BIBO (Bounded Input Bounded Output)

Suppose $|x(n)| < B$ (bounded input)

Under what condition on $h(n)$, the output $y(n)$ remains bounded?

CLAIM: If $\sum_{n=-\infty}^{\infty} |h(n)| = S < \infty$

then LTI system is stable.

Proof:

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h(k)\, x(n-k)| \qquad |A+B| \leq |A| + |B|$$

$$\leq \sum_{k=-\infty}^{\infty} |h(k)| \underbrace{|x(n-k)|}_{< B}$$

$$|y(n)| \leq B \underbrace{\sum_{k=-\infty}^{\infty} |h(k)|}_{= S < \infty}$$

$$|y(n)| \leq S \cdot B < \infty$$

x ——— y

On otherhand, if $\sum_{k=-\infty}^{\infty} |h(k)| = \infty$

then system is unstable

There are bounded inputs $x(n)$ for which output $y(n)$ is unbounded if $\sum_{k} |h(k)| = \infty$

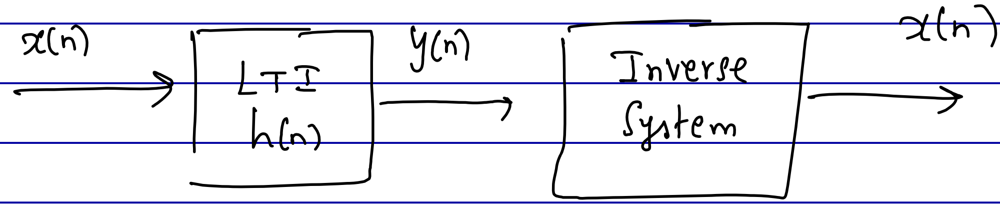
(See Problem 2.49)
in book

In CT case, LTI system with impulse response $h(t)$ is stable if and only if $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$

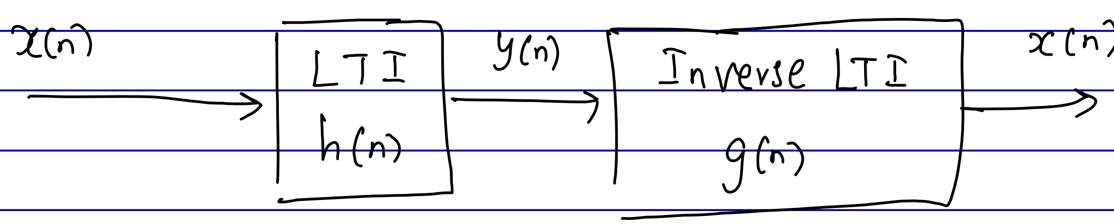
Invertibility of LTI systems.

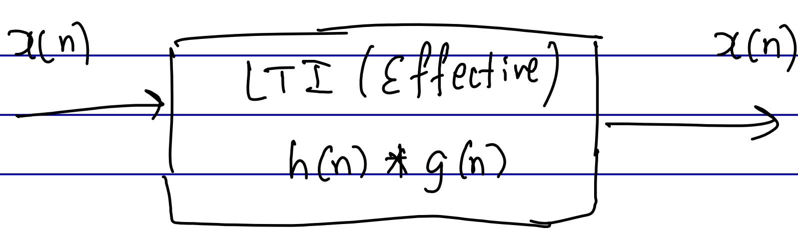
A system is invertible if two distinct input signals produce two distinct output signals.

Equivalently, for invertible systems



For LTI system, the inverse system is also LTI (Problem 2.50 in book)





$$h(n) * g(n) = \delta(n)$$

An LTI system with impulse response $h(n)$ is invertible if there exist a sequence $g(n)$ such that

$$h(n) * g(n) = \delta(n)$$

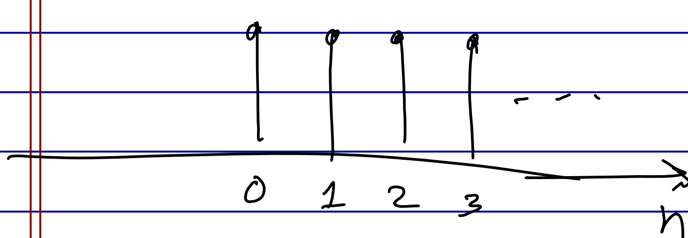
Impulse response of inverse system is $g(n)$.

$x \longrightarrow x$

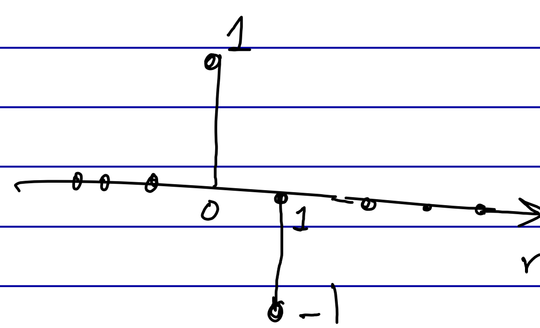
Accumulator.

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

$h(n) = u(n)$



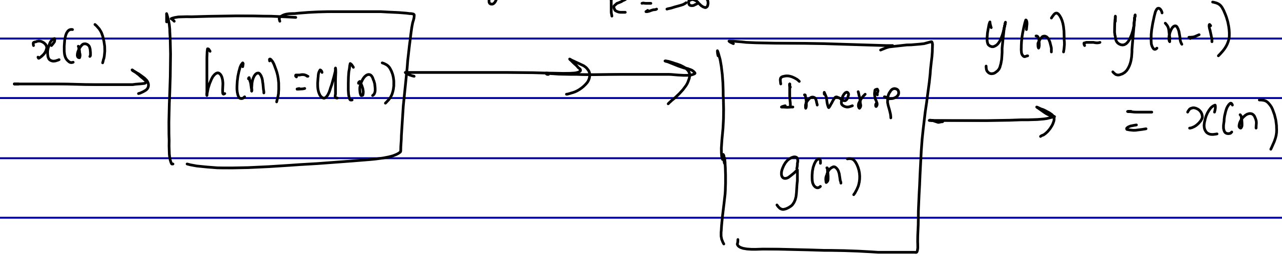
$g(n)$



$$h(n) * g(n) = g(0)\,h(n) + g(1)\,h(n-1)$$

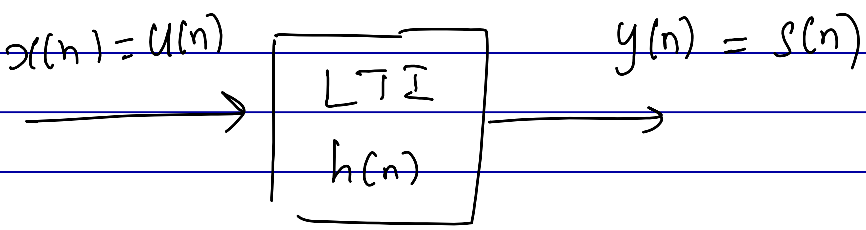
$$= \delta(n)$$

Accumulator $y(n) = \sum_{k=-\infty}^{n} x(n)$



x ———————————————— x

Unit Step response.



$s(n)$ is the unit step response of system when the input to system is $u(n)$

$$s(n) = u(n) * h(n)$$

$$= \sum_{k=-\infty}^{n} h(k)$$

$$h(n) = s(n) - s(n-1)$$

In CT case.

Note Title 03-09-2015

step response $s(t)$

impulse response $h(t)$

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

$$h(t) = \frac{d\, s(t)}{dt}$$