LCCDE

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# Linear Constant Coefficient difference equations.

LCCDE : General form

$$a_0 y[n] + a_1 y[n-1] + \dots + a_N y[n-N]$$

(*) — $$= b_0 x[n] + b_1 x[n-1] + \dots + b_M x[n-M]$$

$a_0, \dots, a_N, b_0, \dots, b_M$ are constants

order/degree of equation $N$

Consider systems with input/output characterized by LCCDE of form (*)

(a) Will they be LTI?

(b) Will they be causal?

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Look at Some Examples first

① Accumulator

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

$$= \sum_{k=-\infty}^{n-1} x(k) + x(n)$$

$$= y(n-1) + x(n)$$

$$y(n) - y(n-1) = x(n)$$

→ Accumulator can be written as LCCDE.

② Moving average

$$y(n) = \frac{1}{M+1} \sum_{k=0}^{M} x(n-k)$$

↓

Average of latest M+1 samples

$\{x(n), x(n-1), \ldots, x(n-M)\}$

With simple algebra,

$$y(n) - y(n-1) = \frac{1}{M+1}[x(n) - x(n-M-1)]$$

Again, we have LCCDE

Does LCCDE <u>always</u> give an LTI system?

<u>No</u>. It depends on initial/auxiliary conditions

Consider general LCCDE

$$\sum_{k=0}^{N} a_k \, y(n-k) = \sum_{m=0}^{M} b_m \, x(n-m) \quad \text{--- } (*)$$

Solution to $(*)$ is of the form

$$y(n) = y_h(n) + y_p(n)$$

$y_h(n) \rightarrow$ homogeneous part which is solution to equation (absence of input)

$$\sum_{k=0}^{N} a_k \, y(n-k) = 0$$

$y_p(n) \rightarrow$ particular solution specific to the applied input $\{x(n)\}$

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To solve homogeneous part

$$\sum_{k=0}^{N} a_k \, y(n-k) = 0 \longrightarrow (\$)$$

Consider solution of form $\alpha z^n$

(Need to find values of $\alpha$ & $z$)

$$\sum_{k=0}^{N} a_k \, \alpha \, z^{n-k} = 0$$

$$\alpha z^n \sum_{k=0}^{N} a_k z^{-k} = 0$$

$$\Rightarrow \underbrace{\sum_{k=0}^{N} a_k z^{-k}}_{\text{Polynomial with } N \text{ roots}} = 0 \quad \text{---} \quad (\#)$$

$\cancel{z}$

Solution to $(\$)$ is of the form

$$y_h(n) = \sum_{l=1}^{N} \alpha_l \, z_l^n$$

where $z_1, z_2, \ldots, z_N$ are roots which satisfy $(\#)$

$\alpha_l$'s are arbitrary

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* LCCDE (in general) does not <u>completely</u> specify the output in terms of input

* Additional information usually refered as auxiliary conditions are needed to find the exact output sequence.

* ———— *

<u>Auxiliary Conditions</u>

can be given by specifying the output $y(n)$ at some time instants.

For instance, $y_h(n)$ has

N unknown constants $(\alpha_1, \ldots, \alpha_N)$

Suppose $\{y(-1), y(-2), \ldots, y(-N)\}$ are given, then exact output of LCCDE can be got

Recursive computation of output:

Suppose $\{y(-1), \ldots, y(-N)\}$ are given.

Rewrite LCCDE as

$$y(n) = -\sum_{k=1}^{N} \frac{a_k}{a_0} y(n-k) + \sum_{m=0}^{M} \frac{b_m}{a_0} x(n-m)$$

Using above <u>forward</u> recursion,

we can find $y(n)$, $\forall$ $\underline{\underline{n \geq 0}}$

We also have

$$y(n-N) = -\sum_{k=0}^{N-1} \frac{a_k}{a_N} y(n-k) + \sum_{m=0}^{M} \frac{b_m}{a_N} x(n-m)$$

Using this <u>backward</u> recursion, we can find output for $n \leq -N-1$

# Auxiliary Condition

If at least one of $\{y(-1), y(-2), \ldots, y(-N)\}$ is non zero, then LCCDE system gives non-zero output even if input is zero $\{x(n) = 0 \;\forall n\}$

* LCCDE with non-zero auxiliary conditions is <u>not</u> LTI

* On the other hand, if $y(-1) = \cdots = y(-N) = 0$ then LCCDE (*) gives an LTI system

* Also, if we impose the <u>initial rest</u> condition, that is

  $y(n) = 0 \;\forall\, n \le n_0$ whenever

  $x(n) = 0 \;\forall\, n \le n_0$

  then LCCDE system is <u>causal</u> & LTI.

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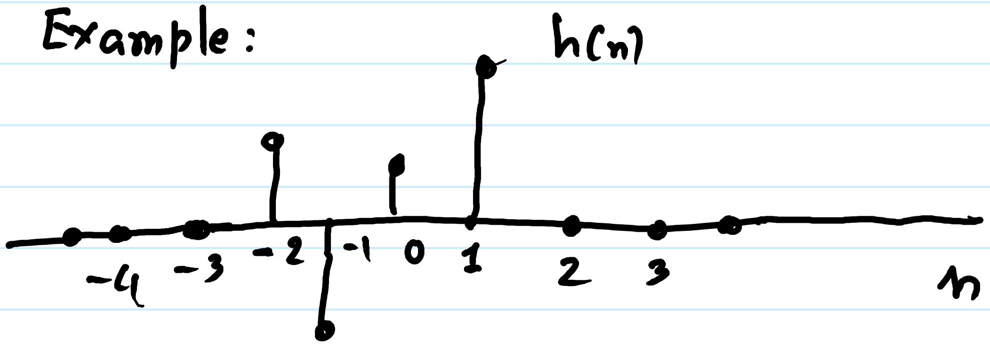
## Finite Impulse Response (FIR)

If impulse response of LTI system exists only for finite time duration, then it is called FIR system.

Example:




$h(n) = 0$ if $n \geq 2$

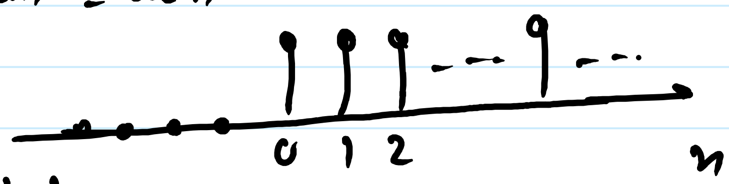
or

if $n \leq -3$

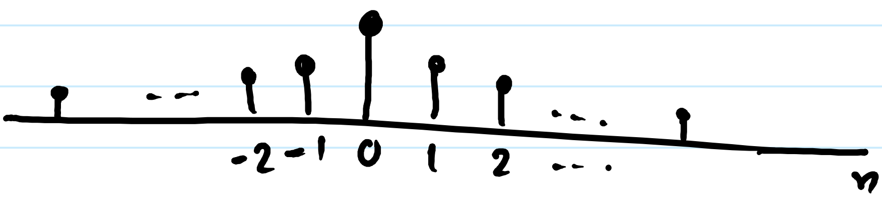
## Infinite Impulse Response (IIR)

If impulse response exists for infinite time duration, then it is an IIR system

Examples: ① $h(n) = u(n)$




② $h(n) = a^{|n|}$, $0 < a < 1$



① Consider LCCDE

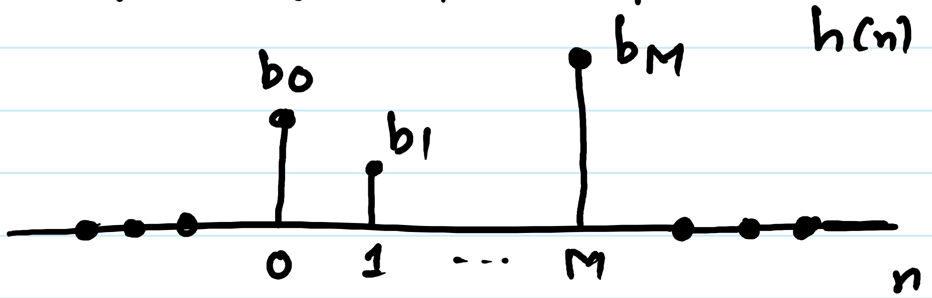
$$y(n) = \sum_{m=0}^{M} b_m \, x(n-m)$$

Causal, LTI

Recall Convolution

$$y(n) = \sum_{m=-\infty}^{\infty} h(m) \, x(n-m)$$

Corresponding impulse response $h(n)$




FIR system

② Consider LCCDE : $y(n) = a\,y(n-1) + x(n)$

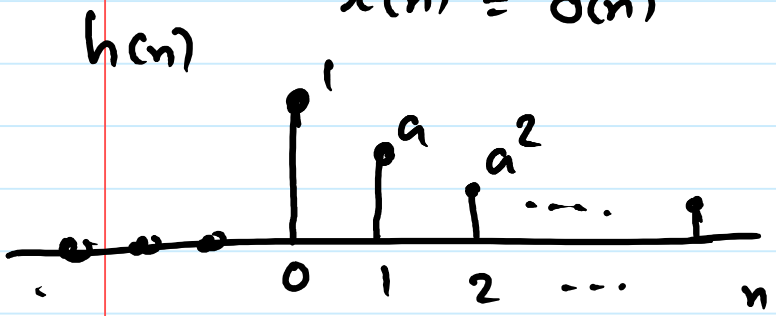
$$0 < a < 1$$

Suppose $y(n) = 0 \quad \forall n < 0$

(Initial rest)

Find impulse response:

$$x(n) = \delta(n)$$




$y(0) = 1$

$y(1) = a$

$\vdots$

$y(20) = a^{20}$

$y(n) = a^{n} u(n)$

(IIR system)