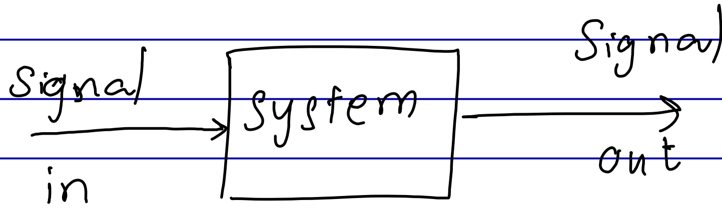
Note Title

EE 2002: Networks & SYSTEMS

03-08-2015

Abstract Model:



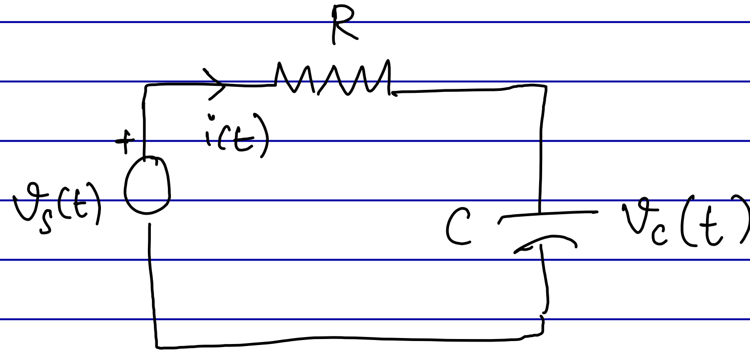
Signal: information bearing quantity

eg: ECG, Traffic signal, signals from cell phone

System: Does some operation / processing on signals (input) and produces another signal (output)

Examples:

(1) Electrical System



Note Title 03-08-2015

$v_s(t) \longrightarrow$ System $\longrightarrow v_c(t)$

Source voltage

capacitor voltage

$t \rightarrow$ tim variable

EMC course tells you how to analyse this electrical system

(2) Financial system

deposit/withdraw $\longrightarrow$ Bank $\longrightarrow$ Account balance

$\downarrow$

$x[n]$

amount of money deposited at time n

$y(n)$

$\downarrow$

balance at time n

Here time n takes only discrete values

Note Title 03-08-2015

③ Communication System

Speech signal (acoustic pressure) $\longrightarrow$ Cell phone $\longrightarrow$ Signal sent from antenna (electromagnetic wave)

Main Theme

- As long as systems (which may look different) satisfy some mathematical properties, they can be studied under a common mathematical framework

- This course develops powerful mathematical tools to study a class of systems (linear and time-invariant)

Note Title 4/8 Classification of 04-08-2015

Signals / Systems

① Continuous-time (CT) Signals / Systems

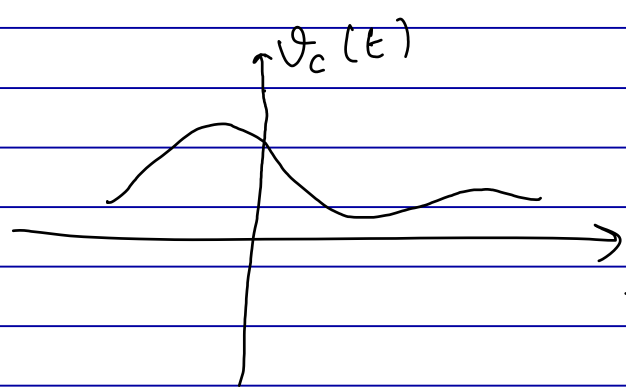
② Discrete-time (DT) Signals / Systems

CT Signal

A signal specified for continuum of values of time t is CT signal

examples: Many physical signals are CT signals

for instance $v_c(t) \rightarrow$ voltage across a capacitor



Independent variable t takes continuum of values

In some situations, t may denote space, height, distance etc (not necessarily time)

Note Title 04-08-2015

A system which operates on CT signals is a continuous-time system

A Discrete-time (DT) signal is specified only for discrete values of time

examples: real world signals
(bank account balance at end of each day)

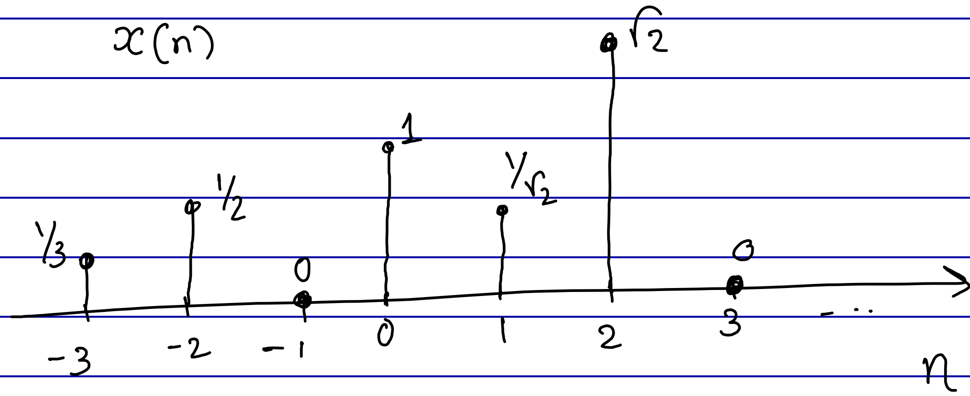
- In this course, we consider the discrete values of time to be integers.

Let $x[n]$ be a discrete-time signal

$x(n)$ is defined only for $n \in \mathbb{Z}$

Set of integers

If $n \notin \mathbb{Z}$, $x(n)$ is <u>undefined</u>



$n \rightarrow$ discrete - time index

(takes only integer values)

A DT system takes DT signal as input and produces DT signal as output.

Note Title 04-08-2015

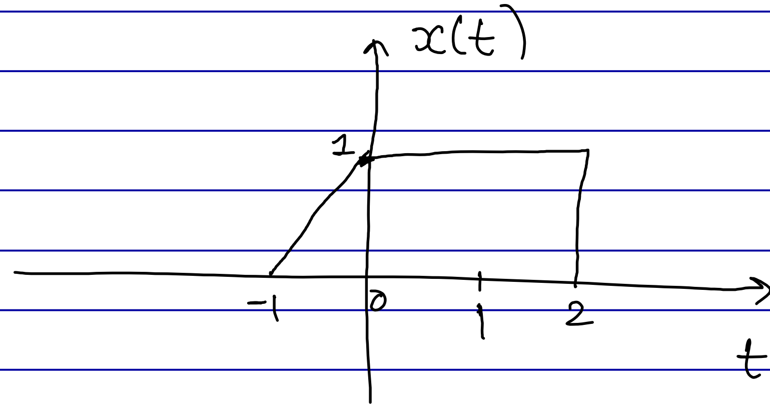
Basic Signal Transformations
(on independent time variable)

1* Translation or time shifting

2* Time scaling

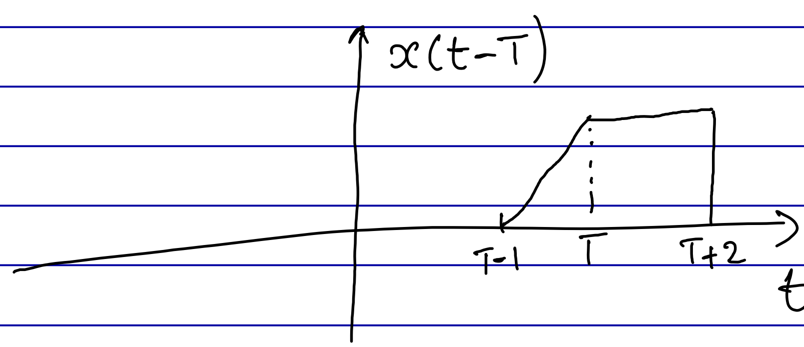
3* Time scaling <u>and</u> time shifting

Time shifting:



A Signal $x(t)$ is time shifted by T seconds by replacing t with $t-T$.

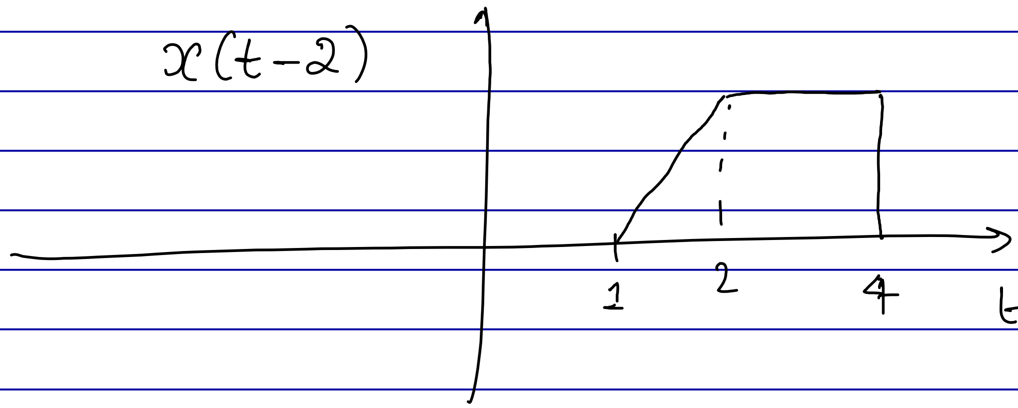
$x(t-T)$ is time shifted version of $x(t)$, shifted by T seconds



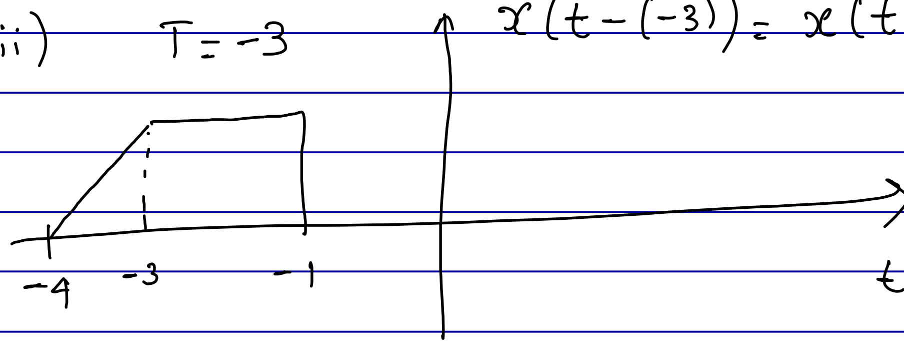
Note Title 04-08-2015

Case (i) $\tau = 2$

$x(t-2)$



case (ii) $\tau = -3$ $x(t-(-3)) = x(t+3)$



for $\tau > 0$; $x(t-\tau)$ is time delayed version of $x(t)$

For $\tau < 0$; $x(t-\tau)$ is time advanced version of $x(t)$

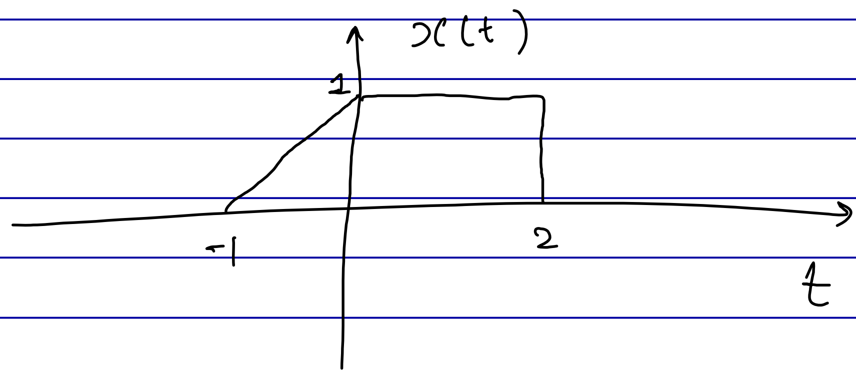
Note Title 04-08-2015

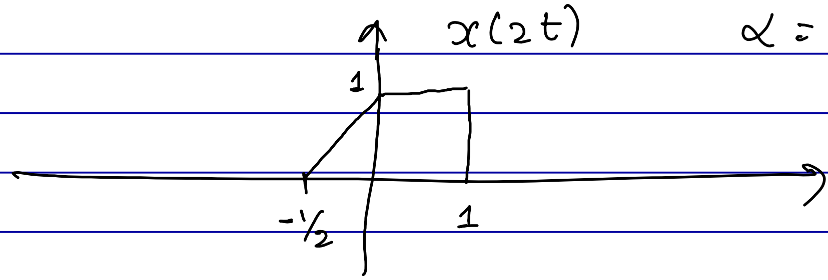
* Time scaling

Multiply the independent time variable t by a real constant α

$$y(t) = x(\alpha t)$$

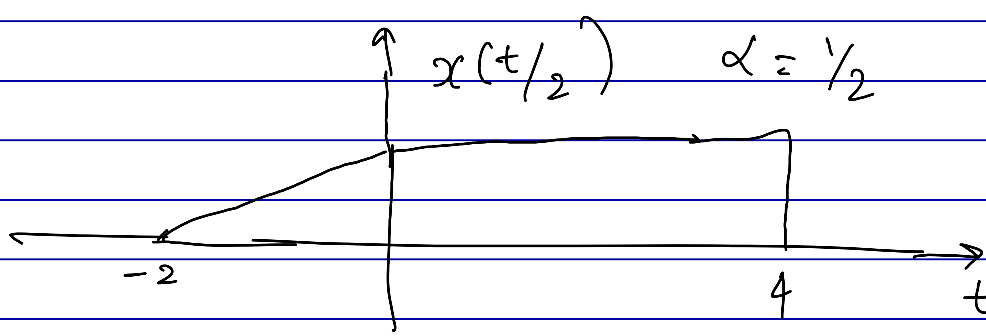
time scaling leads to expansion or compression in time





$\alpha = 2$

$\alpha > 1$ then it leads to compression in time



$\alpha = 1/2$

Note Title 5/8 Transformations on independent Variable. 05-08-2015

→ Time Scaling

$x(\alpha t)$ is time scaled version of signal $x(t)$

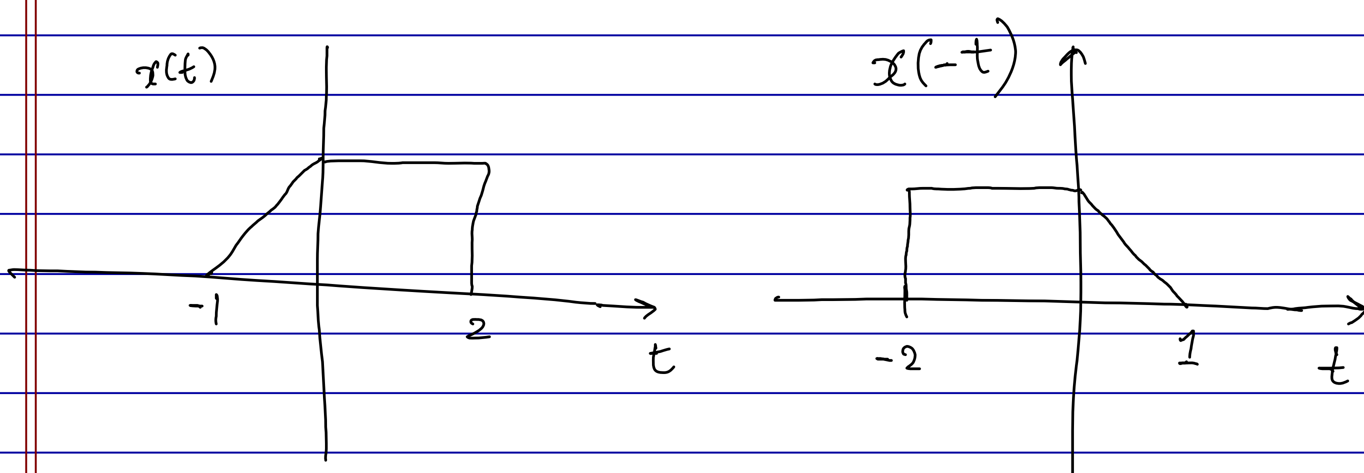
α is a real constant

$\alpha > 1 \rightarrow$ Compression

$0 < \alpha < 1 \rightarrow$ Expansion

What about negative values of α ?

Consider $\alpha = -1$

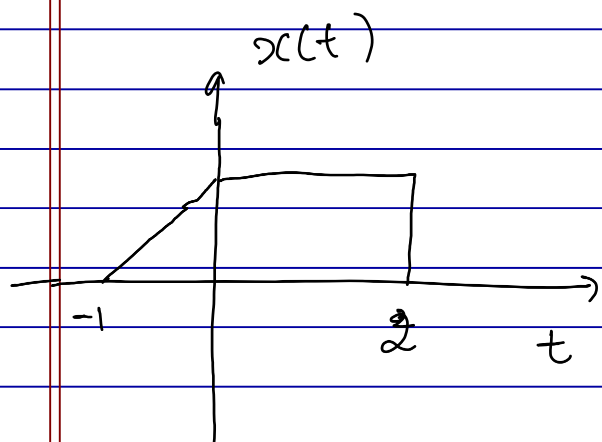


$x(-t)$ is time reversed version of $x(t)$

Note Title 05-08-2015

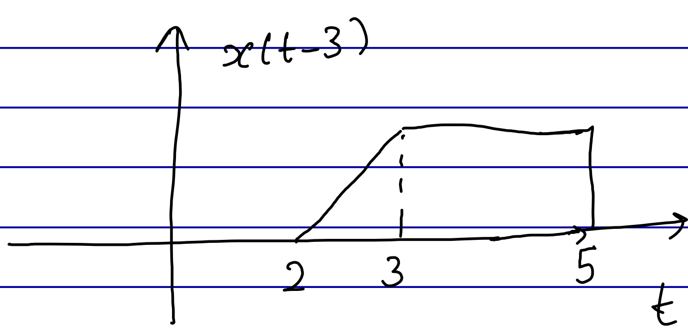
* Combination of time scaling and time shifting

$x(at - b) \rightarrow$ how is it related to $x(t)$

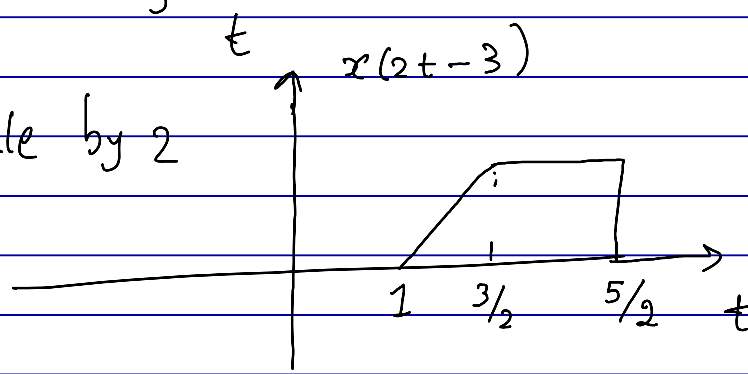


Want to find
$x(2t-3) = y(t)$

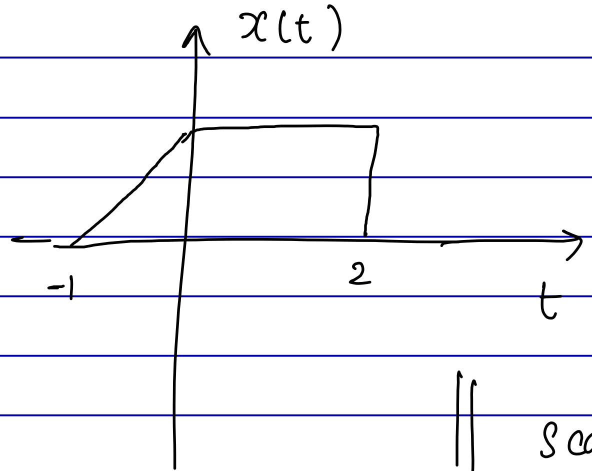
shift by 3
$x(t-3)$



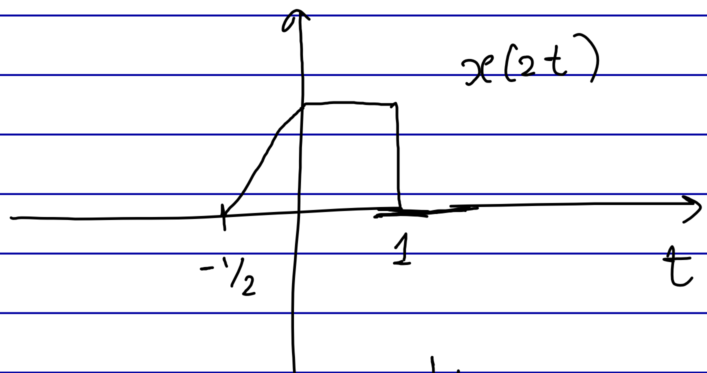
scale by 2



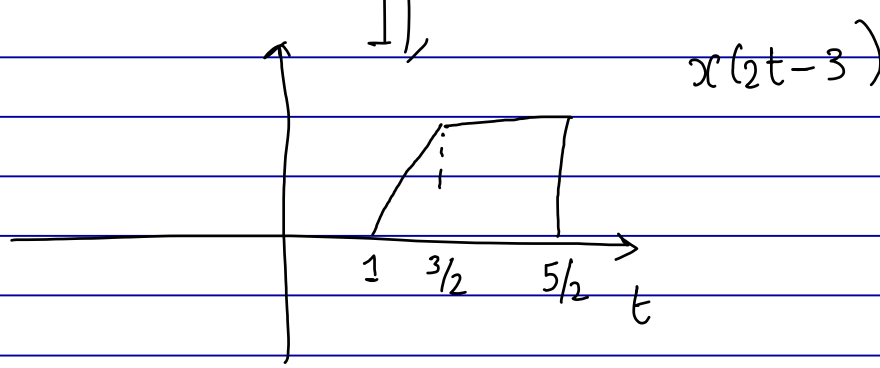
Note Title 05-08-2015



$\Downarrow$ scale by factor of 2



$\Downarrow$ shift

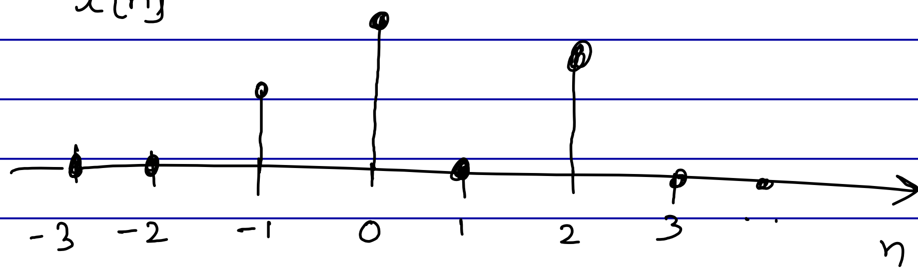


$y(t) = x(at-b)$

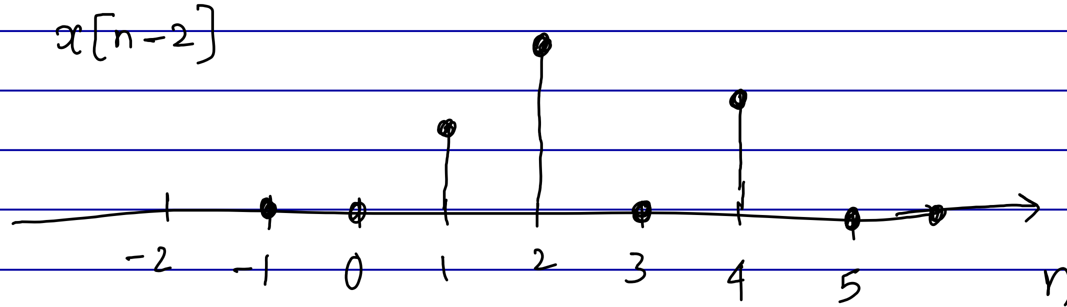
$$
\begin{tikzcd}
x(t) \arrow[r, "\text{shift by } b"] \arrow[d, "\text{scale by } a"] & x(t-b) \arrow[d, "\text{scale by } a"] \\
x(at) \arrow[r, "\text{shift by } b/a"] & x(at-b)
\end{tikzcd}
$$

Time Shifting & Scaling for DT Signals

$x[n]$



$x[n-2]$



DT time shifting is very Similar to CT time shifting

- Time scaling

$$y[n] = x[\alpha n]$$

If α is an integer, $y[n]$ is well defined.

Note Title 05-08-2015

- $\alpha = 2$ $y[n] = x(2n)$

$\downarrow$

contains the even time samples of $x(n)$

- $\alpha = 1/2$ $y(n) = x(n/2)$

$\Downarrow$

$y(n)$ is not defined for odd values of n

- $\alpha = -1$ $y(n) = x(-n)$

$\Downarrow$

$y(n)$ is time reversed version of $x(n)$

Note Title 05-08-2015

Basic Properties of Signals

1) Periodic and Aperiodic Signals

Signal $x(t)$ is called <u>periodic</u> if there exist a non-zero constant T such that

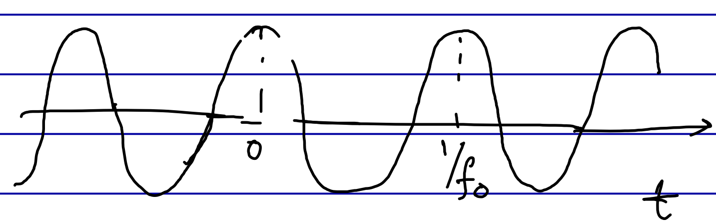
$$(*) \longrightarrow \underbrace{x(t)}_{\text{original signal}} = \underbrace{x(t+T)}_{\text{time shifted signal}} \quad (\text{for all } t)$$

The smallest value of T for which above condition $(*)$ holds is called fundamental period T_0

Fundamental frequency is the inverse of the fundamental period.

Example;

$x(t) = \cos(2\pi f_0 t)$



Note Title 05-08-2015

Fundamental period $T_0 = 1/f_0$

$$x(t+T_0) = \cos\left(2\pi f_0 (t+T_0)\right)$$

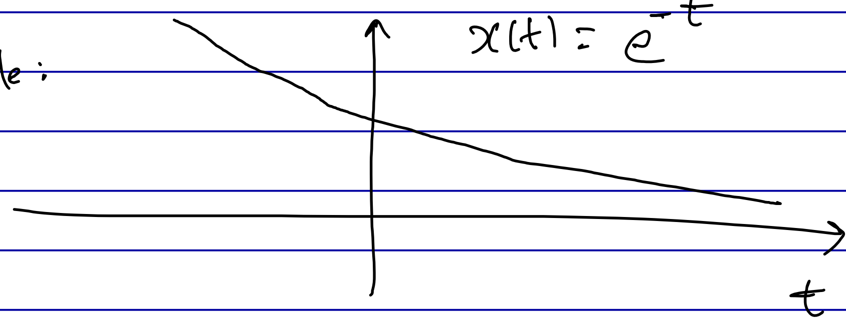
$$= \cos\left(2\pi f_0 t + 2\pi f_0 \cdot \frac{1}{f_0}\right)$$

$$= \cos\left(2\pi f_0 t + 2\pi\right)$$

$$= \cos(2\pi f_0 t) = x(t)$$

Signals which are <u>not periodic</u> are called aperiodic signals.

Example: $x(t) = e^{-t}$



Similarly define periodic DT Signals:

If there exist non-zero integer N such that $x[n] = x[n+N]$ then DT signal/Sequence $x[n]$ is periodic.

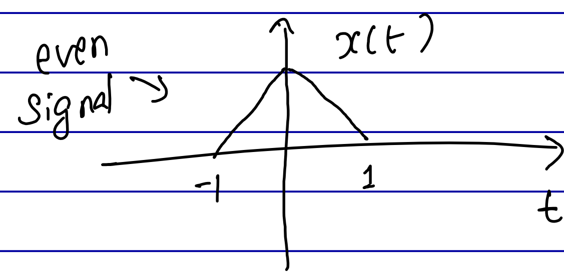
Even and Odd Signals

* A Signal $x(t)$ is called even signal if $x(t) = x(-t)$ (for all t)

(original signal is identical to time reversed signal)

Example:

$\cos(2\pi f_0 t)$

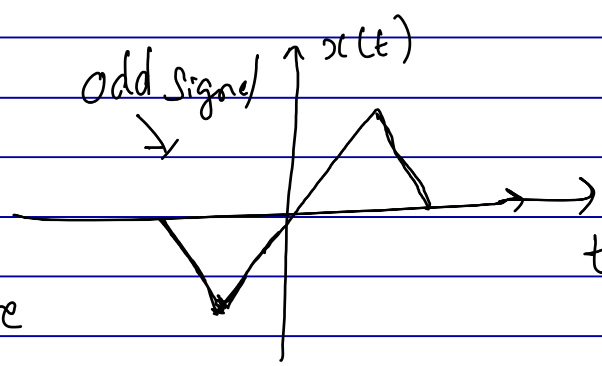


even signals are symmetric about time $t=0$

* A Signal $x(t)$ is odd if $x(t) = -x(-t)$ (for all t)

Examples:

$\sin(2\pi f_0 t)$



(odd signals are anti-symmetric about time $t=0$)

Note Title 06-08-2015

Note: Any Signal $x(t)$ can be written as a sum of two parts/signal with one of them being an odd signal and other being an even signal

Consider arbitrary signal $x(t)$

Define $$x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

Easy to verify that

$x_e(t)$ is an even Signal

$x_o(t)$ is odd signal

$$x(t) = x_e(t) + x_o(t)$$

Same definitions apply to DT signals also

$x[n] = x[-n] \rightarrow$ even DT signal

$x[n] = -x[-n] \rightarrow$ odd

ENERGY & POWER of Signals.

Note Title 06-08-2015

- Let $x(t)$ be a voltage signal across a resistor R in an electrical network.

power = current × voltage

$$= \frac{\text{voltage}}{R} \times \text{voltage}$$

$$= x(t)^2 / R$$

Energy dissipated in interval $[T_1, T_2]$

$$= \int_{T_1}^{T_2} \frac{x^2(t)}{R}\, dt$$

Let us generalize the notion of power and energy for arbitrary signals

- Let $x(t)$ be a general complex-valued signal

defined in interval $-\infty < t < \infty$

Note Title 06-08-2015

Energy of Signal $x(t)$ is defined as

$$E_\infty = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt$$

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Power of Signal $x(t)$ is defined as

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

From these definitions, it follows that

If E_∞ is finite then $P_\infty = 0$

If $P_\infty > 0$ then $E_\infty = \infty$

Example : $x(t) = \cos(t)$

Here $E_\infty = \infty$

$P_\infty = ?$

× ———————— ×

Note Title

Two Important Signals

06-08-2015

CT & DT
① Complex Exponentials
② Unit impulse & Unit Step

× ——— ×

Continuous-time Complex Exponentials

In the most general form

$$x(t) = A e^{st}$$

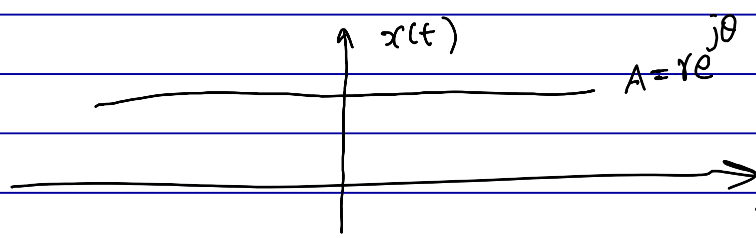
A and s are complex numbers (in general)

- s is called complex frequency

Suppose $s = \sigma + j\omega$ $\quad A = r e^{j\theta}$

Consider the following special cases

① $s = 0$ $\quad x(t) = A e^{0 \cdot t} = A$



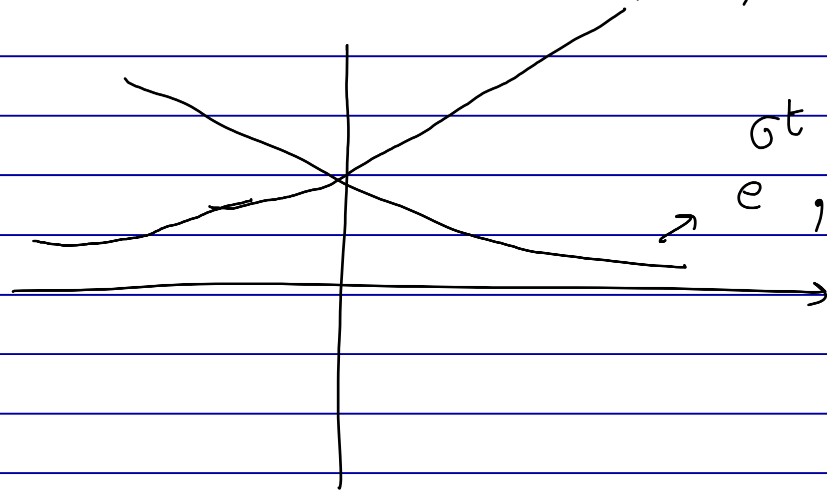
This is called a DC signal

Note Title 06-08-2015

(2) $s = \sigma + j\omega$

Consider $\boxed{\omega = 0}$ and A is real

$x(t) = Ae^{\sigma t}$ & $e^{\sigma t}$, $\sigma > 0$



$e^{\sigma t}$, for $\sigma < 0$

10/8

(3) Consider $Ae^{st} + A^* e^{s^* t}$

$()^* \rightarrow$ denotes conjugation

$$= re^{j\theta} \cdot e^{(\sigma + j\omega)t} + re^{-j\theta} e^{(\sigma - j\omega)t}$$

$$= re^{\sigma t} \left(\underbrace{e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)}}_{2\cos(\omega t + \theta)} \right)$$

$$= 2r e^{\sigma t} \cos(\omega t + \theta)$$

$\sigma t \rightarrow$ real part of complex frequency

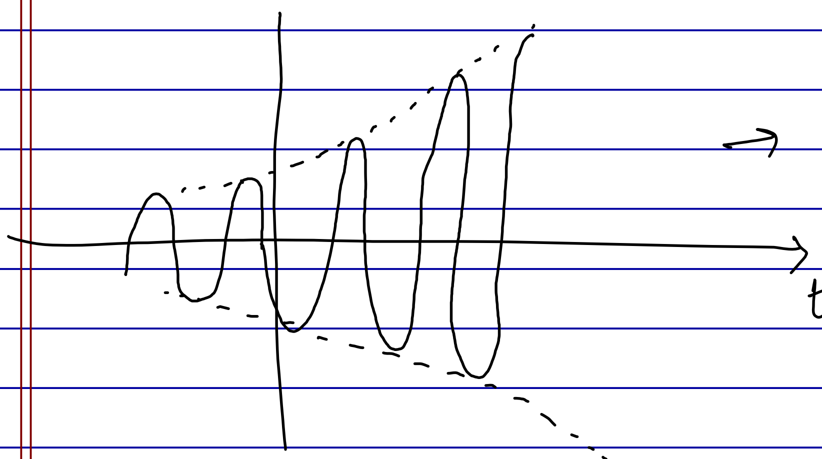
$\omega \rightarrow$ imaginary part of complex freq.

Complex frequencies $\sigma \pm j\omega$

Note Title 10-08-2015

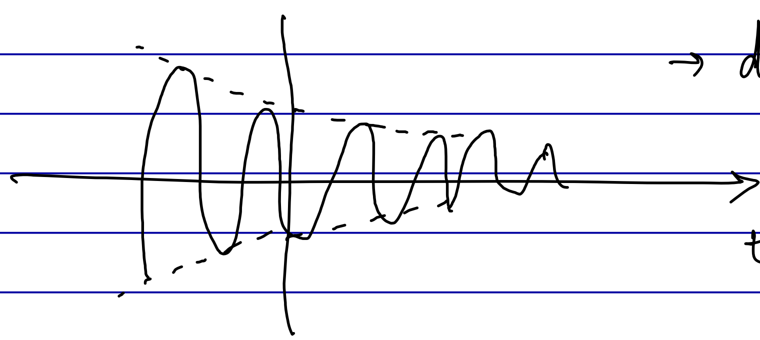
Plots of complex exponentials

① $\sigma \pm j\omega$; $\sigma > 0$ $\quad 2r e^{\sigma t} \cos(\omega t + \theta)$



→ growing

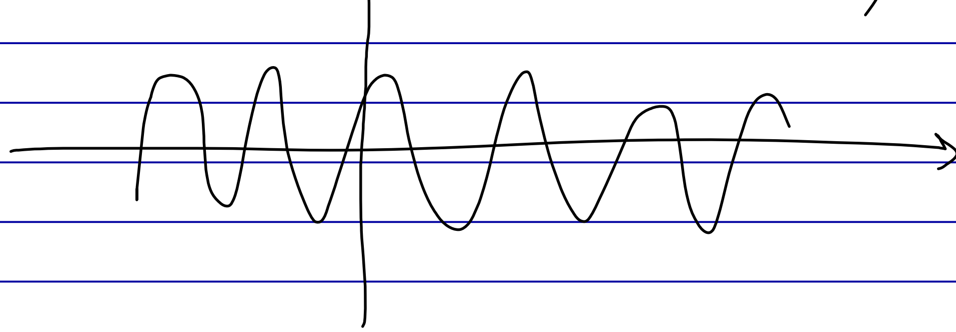
② $\sigma \pm j\omega$; $\sigma < 0$



→ decaying

③ $\pm j\omega$; $\sigma = 0$

$2r \cos(\omega t + \theta)$



Note Title 10-08-2015

Signal	Complex frequencies
① $\cos(2t+1)$	$\pm 2j$
② $\sin(3t+5)$	$\pm 3j$

$$\sin(3t+5) = \frac{e^{j(3t+5)} - e^{-j(3t+5)}}{2j}$$

$$= \frac{e^{j5}}{2j} e^{j3t} - \frac{e^{-j5}}{2j} e^{-j3t}$$

③ $4 + 3e^{2t}\sin(5t-1)$

Complex frequencies

$0,\ 2 \pm 5j$

④ $e^{2jt} + e^{-4t}\cos(5t+1) - \sin(3t+2)$

$+2j,\ -4 \pm 5j,\ \pm 3j$

x ———— x

Note Title 10-08-2015

When $s = j\omega$ $(\sigma = 0)$

$$Ae^{st} = Ae^{j\omega t}$$

$$= re^{j\theta} e^{j\omega t}$$

Say $x(t) = re^{j(\omega t + \theta)}$

- $x(t)$ is periodic with fundamental period $= \frac{2\pi}{\omega}$

- Power of $x(t)$ $P_\infty = r^2$

- Energy of $x(t)$ $E_\infty = \infty$

x ——————— x

Note Title 10-08-2015

DT Complex exponential

$$x[n] = e^{j\omega_0 n}, \quad n \in \mathbb{Z}$$

ω_0 is a constant

- For ω_0 & $\omega_0 + 2\pi k$ the DT complex exponential is identical for any integer k.

$$e^{j(\omega_0 + 2\pi k)n} = e^{j\omega_0 n} \cdot \underbrace{e^{j2\pi k n}}_{1}$$

$$= e^{j\omega_0 n}.$$

- $x(n) = e^{j\omega_0 n}$ is periodic only if it satisfies certain condition

Suppose $x(n) = x(n+N)$ N is an integer

$$e^{j\omega_0 n} = e^{j\omega_0 (n+N)}$$

$$= e^{j\omega_0 n} \cdot e^{j\omega_0 N}$$

from original signal

Note Title 10-08-2015

$\Rightarrow e^{j\omega_0 N} = 1$

$\Rightarrow \omega_0 N = 2\pi m$ for some integer m

$\Rightarrow \boxed{\omega_0 = 2\pi \frac{m}{N}}$ for some integers m & N

$\Downarrow$

Condition when DT complex exponential is periodic.

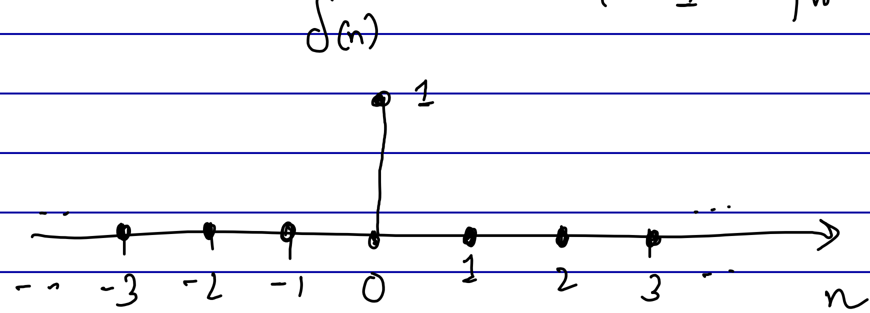
$\omega_0 = 1$ will <u>not</u> <u>be</u> <u>periodic</u>.

Unit Impulse & Unit Step functions

Start with DT case.

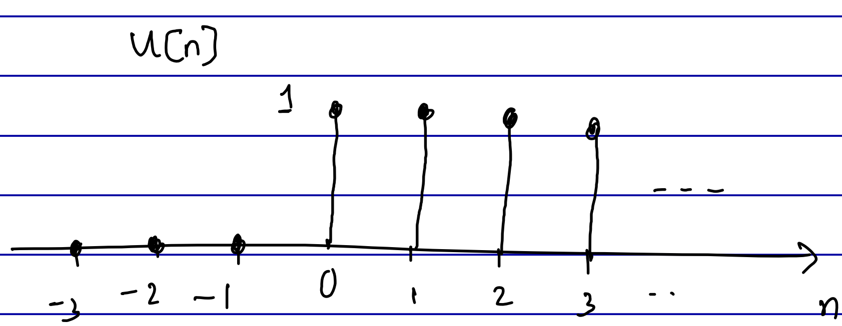
DT unit impulse signal $\delta(n)$ is defined as

$$\delta(n) = \begin{cases} 0 & \text{for } n \neq 0 \quad n \in \mathbb{Z} \\ 1 & \text{for } n = 0 \end{cases}$$



DT unit step signal $u[n]$ is

$$u[n] = \begin{cases} 0 & \text{for } n < 0 \quad \& \\ 1 & \text{for } n \geq 0 \quad n \in \mathbb{Z} \end{cases}$$



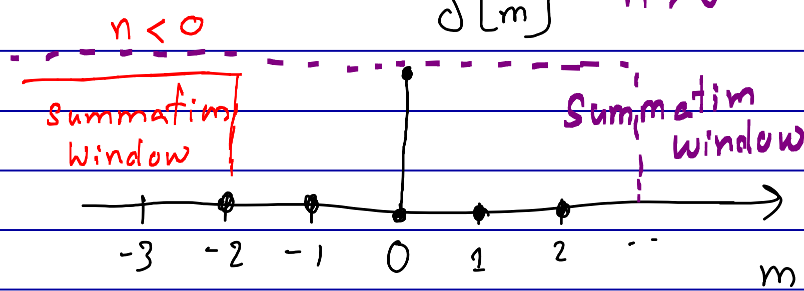
Note that $\delta(n) = u(n) - u(n-1)$

& can be thought as substracting time shifted signal

Note Title 11/8 11-08-2015

Writing step function $u[n]$ using delayed copies of $\delta(n)$ (Summation)

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$



Summation window is $-\infty$ to n

$$u(n) = \sum_{m=-\infty}^{n} \delta[m]$$

change of variable $k = n - m$

$$u(n) = \sum_{k=0}^{\infty} \delta(n-k)$$

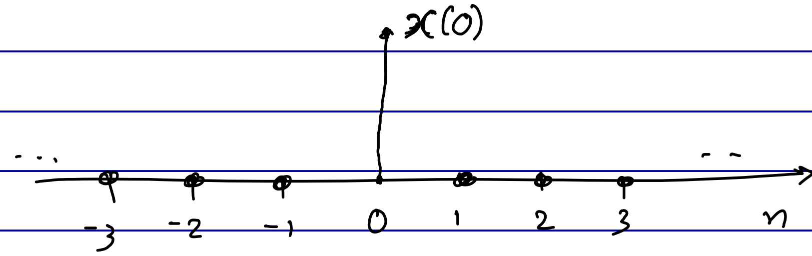
$$= \delta(n) + \delta(n-1) + \delta(n-2) + \cdots$$

(infinite) summation of delayed copies of $\delta(n)$

Sampling Property of $\delta(n)$

Let $x[n]$ be an arbitrary DT Signal

Consider $x[n]\,\delta[n] \rightarrow$ product of $x[n]$ and $\delta[n]$



$$x(n)\,\delta(n) = x(0)\,\delta(n)$$

$$x(n)\,\delta(n-N) = x(N)\,\delta(n-N)$$

$$x \longrightarrow x$$

CT unit step function

$u(t) \rightarrow$ CT unit step function is defined as

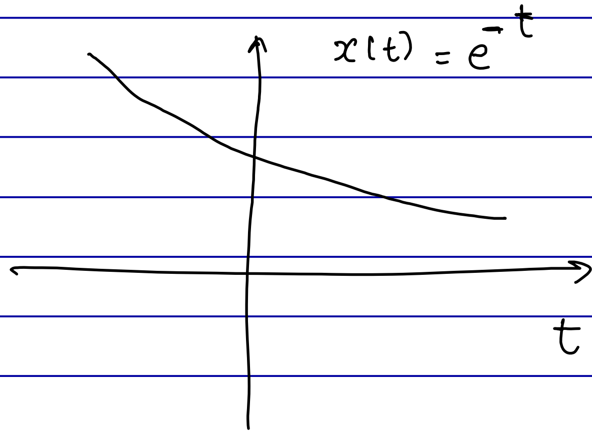
$$u(t) = \begin{cases} 0 & ; \text{ for } t < 0 \\ 1 & ; \text{ for } t \geq 0 \end{cases}$$

Using $u(t)$, we can compactly represent some types of Signals.

Example

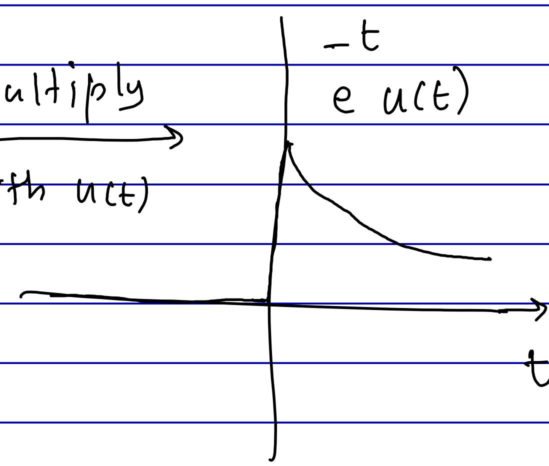
1. Suppose we want to nullify signal for negative values of t, we can simply multiply by u(t)

$x(t) = e^{-t}$

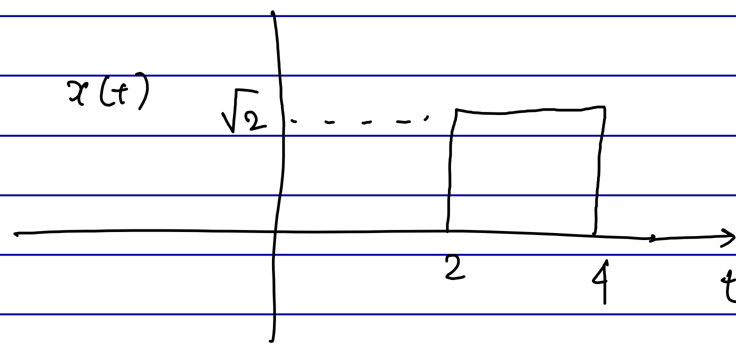


Multiply with u(t) $\longrightarrow$

$e^{-t}u(t)$

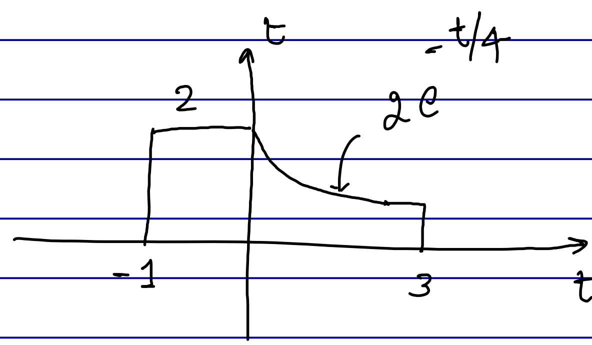


2. $x(t)$



$$x(t) = \sqrt{2}\,[u(t-2) - u(t-4)]$$

3.



$$x(t) = \begin{cases} 2 \,; & -1 < t \le 0 \\ 2e^{-t/4} \,; & 0 < t \le 3 \\ 0 \,; & \text{else.} \end{cases}$$

$$x(t) = 2\,[u(t+1) - u(t)] + 2e^{-t/4}\,[u(t) - u(t-3)]$$

CT Impulse function

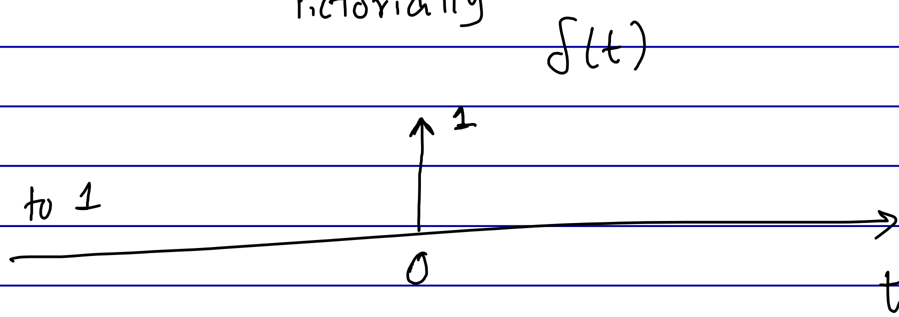
- Need to be careful in defining CT impulse
- We will understand it from its properties

CT impulse function $\delta(t)$ satisfies

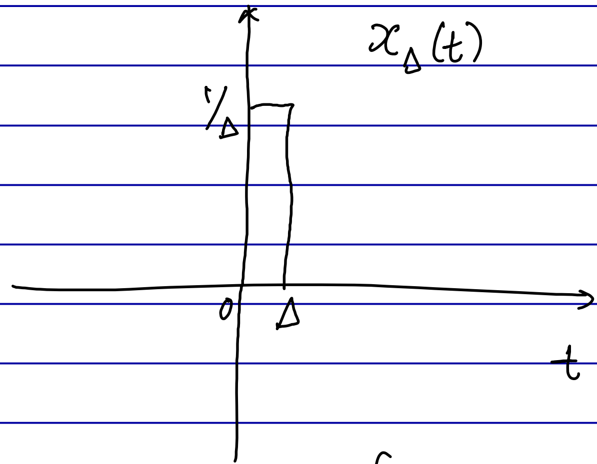
1. $\delta(t) = 0$ for $t \neq 0$
2. $\displaystyle\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

Pictorially

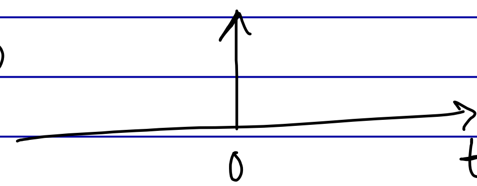
At $t=0$, $\delta(t)$ is not equal to 1



- We can think $\delta(t)$ as a limit of a sequence of functions.



$\lim_{\Delta \to 0}$

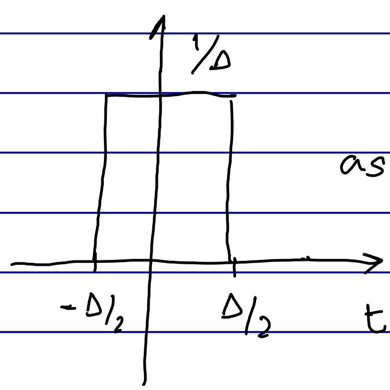


$$x_\Delta(t) = \begin{cases} 1/\Delta & \text{for } 0 \le t \le \Delta \\ 0 & \text{else} \end{cases}$$

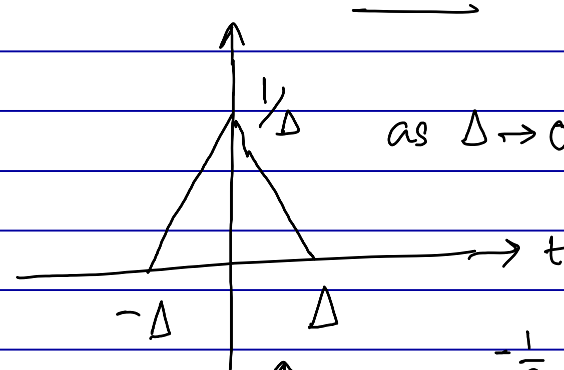
$$\int_{-\infty}^{\infty} x_\Delta(t)\, dt = 1 \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$\delta(t) = 0 \text{ for } t \neq 0$$

12/8 $\delta(t)$ can be obtained thru other limits also



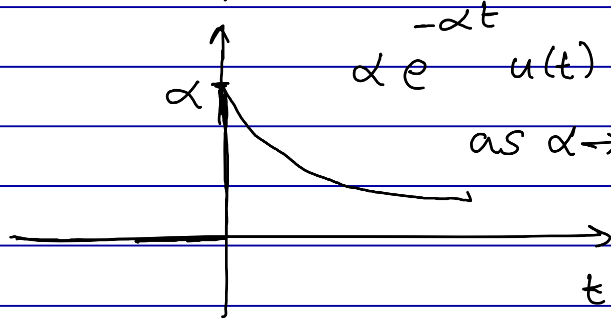
as $\Delta \to 0$



as $\Delta \to 0$

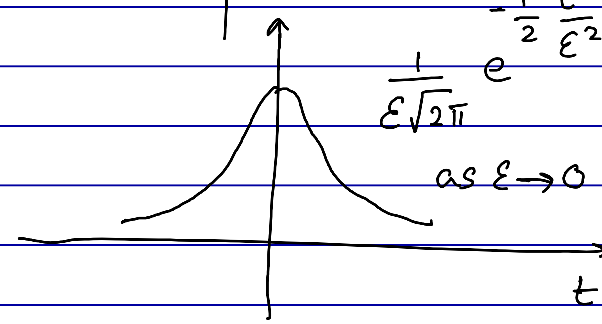
$\alpha e^{-\alpha t} u(t)$

as $\alpha \to \infty$



$\frac{1}{\varepsilon\sqrt{2\pi}} e^{-\frac{1}{2}\frac{t^2}{\varepsilon^2}}$

as $\varepsilon \to 0$



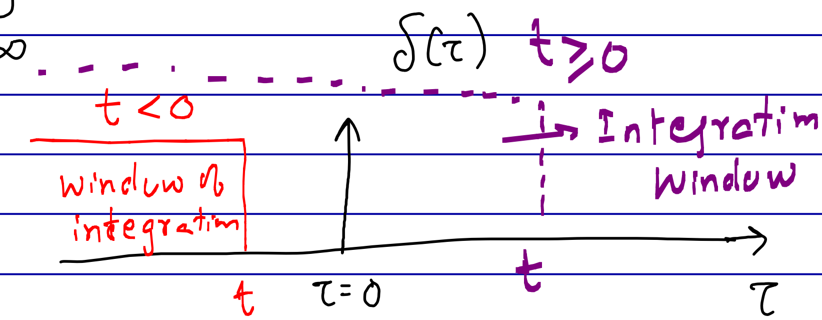
CT step function can be got from CT impulse using integration.

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Consider

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

Integration window $-\infty$ to t



Change of variables

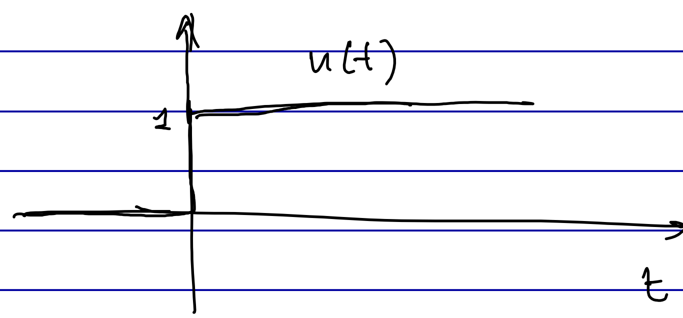
$$\sigma = t - \tau$$

$$u(t) = \int_{\sigma=0}^{\infty} \delta(t-\sigma)\, d\sigma.$$

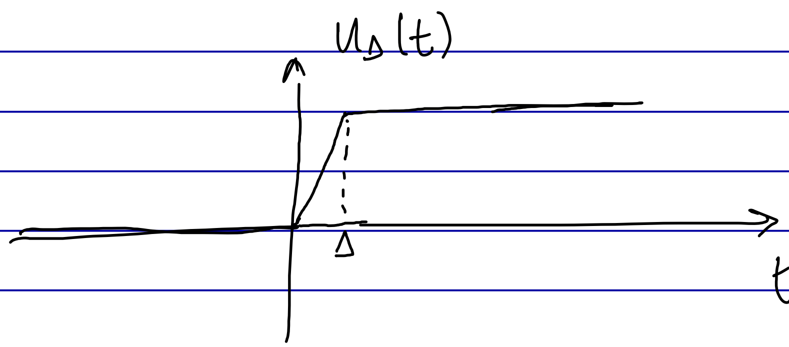
interpreted as "Sum" of delayed copies of impulse function.

Getting $\delta(t)$ from $u(t)$ can be done thru differentiation

But we need to be careful since $u(t)$ is discontinuous at $t=0$



Consider $U_\Delta(t)$

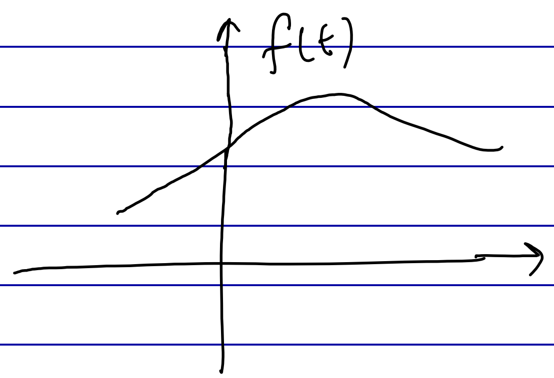


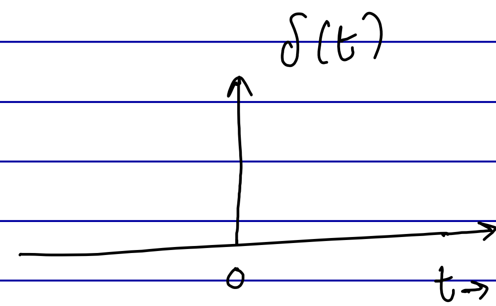
$$\delta(t) = \lim_{\Delta \to 0} \frac{d}{dt} U_\Delta(t)$$

Sampling Property of $\delta(t)$

Let $f(t)$ be some arbitrary function

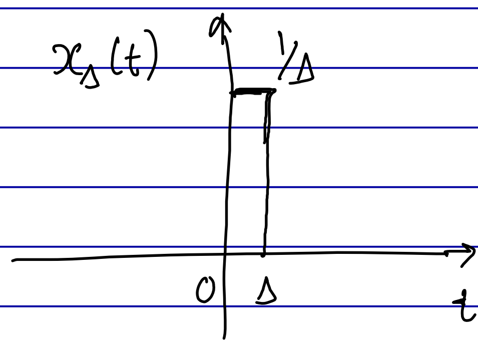
$f(t)$ be continous at $t=0$.





Consider $f(t)\,\delta(t) = ?$

Let $x_\Delta(t)$



Consider $f(t) \cdot x_\Delta(t)$ and then take limit as $\Delta \to 0$

Note Title 12-08-2015

$$f(t)\, x_\Delta(t) = \begin{cases} f(t) \cdot \frac{1}{\Delta} & \text{for } 0 \le t \le \Delta \\ 0 & \text{otherwise} \end{cases}$$

if Δ is very small

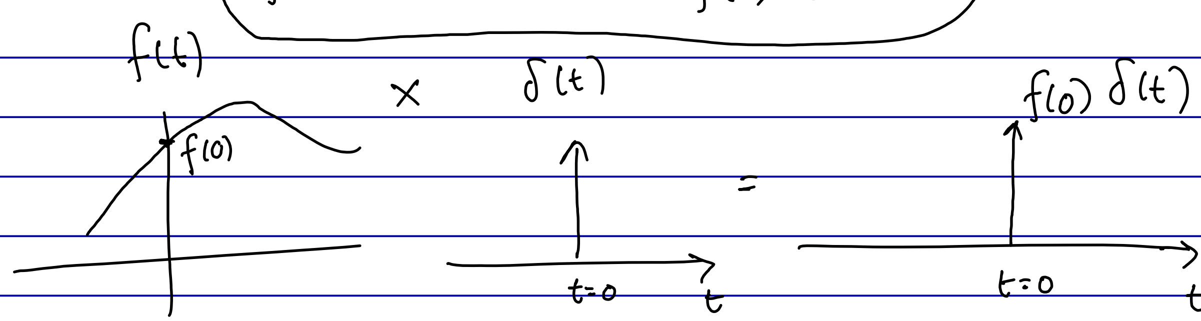
$$\approx \begin{cases} f(0) \cdot \frac{1}{\Delta} & \text{for } 0 \le t \le \Delta \\ 0 & \text{otherwise.} \end{cases}$$

$f(t) \approx f(0)$ for $0 \le t \le \Delta$

$$f(t)\, x_\Delta(t) \approx f(0)\, x_\Delta(t) \quad \text{for very small } \Delta$$

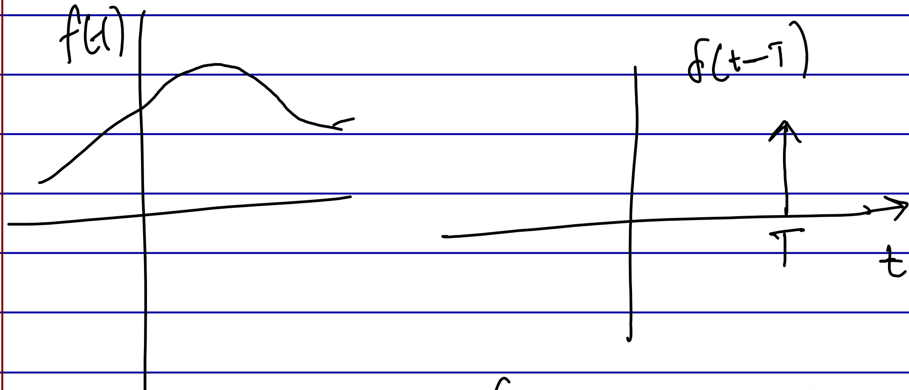
$$\lim_{\Delta \to 0} f(t)\, x_\Delta(t) = \lim_{\Delta \to 0} f(0)\, x_\Delta(t)$$

$$f(t)\,\delta(t) = f(0)\,\delta(t)$$



Similarly;

$$f(t)\,\delta(t-T) = f(T)\,\delta(t-T)$$



(assuming $f(t)$ is continuous at time $t = T$)

Sifting Property of Impulse function.

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-T)\,dt = \int_{-\infty}^{\infty} f(T)\,\delta(t-T)\,dt$$

$$= f(T) \underbrace{\int_{-\infty}^{\infty} \delta(t-T)\,dt}_{=1}$$

$$\boxed{\int_{-\infty}^{\infty} f(t)\,\delta(t-T)\,dt = f(T)}$$

Sifting property.