dtft

06 February 2018 09:49

Discrete time Fourier Transform

$x(n)$ → discrete time signal/sequence

- what are the "frequency components" present in signal?
- why do we need to know the frequency contents?

Discrete time Complex exponential (DTCE)

Recall

$$e^{j\omega n}, \quad n \in \mathbb{Z}$$

ω → angular frequency

$\omega \in [-\pi, \pi]$ → range of interest for angular freq.

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~~Eigen~~ Eigen functions of LTI Systems

Consider LTI System with impulse response $h(n)$

$$x(n) \longrightarrow \boxed{\begin{matrix} \text{LTI} \\ h(n) \end{matrix}} \longrightarrow y(n)$$

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

Suppose input $x(n)$ is DTCE

$$x(n) = e^{j\omega_0 n} ,$$

$\omega_0 \rightarrow$ angular frequency

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega_0 (n-k)}$$

$$= e^{j\omega_0 n} \underbrace{\sum_{k=-\infty}^{\infty} h(k) e^{-j\omega_0 k}}_{H(e^{j\omega_0})}$$

$H(e^{j\omega_0}) \rightarrow$ complex gain

$$= H(e^{j\omega_0})\, x(n)$$

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If input to LTI system is DTCE, output is scaled version of the same DTCE.

The scaling factor - complex gain

$H(e^{j\omega_0})$ depends on

both input frequency ω_0 &

impulse response $h(n)$

Can we break-down an arbitrary sequence $x(n)$ as weighted sum of DTCE ?

Yes, if $x(n)$ satisfies certain conditions, using DTFT.

DTFT Relationship.

$x(n) \rightarrow$ discrete time Signal

We have

inverse DTFT $\rightarrow$ $$x(n) = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}}_{\text{Integration or Summation}} \underbrace{X(e^{jw})}_{\text{weights}}\, \underbrace{e^{jwn}}_{\text{DTCE}}\, dw$$

where

DTFT $\rightarrow$ $$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

$$w \in \mathbb{R}$$

$w \rightarrow$ continuous variable

- $X(e^{jw})$ is called spectrum of signal $x(n)$
- $X(e^{jw})$ is an equivalent representation of signal $x(n)$

Notation

$$x(n) \xleftrightarrow{\mathcal{F}} X(e^{jw})$$

$n \in \mathbb{Z}$

- $x(n) \rightarrow$ Discrete time Signal
- $X(e^{j\omega}) \rightarrow$ function of continuous Variable

$\omega \in \mathbb{R}$

- $X(e^{j\omega})$ is periodic function of ω.

(period is 2π)

$$X(e^{j(\omega+2\pi)}) = \sum_n x(n) e^{-j(\omega+2\pi)n}$$

$$= \sum_n x(n) e^{-j\omega n}$$

$$= X(e^{j\omega})$$

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"Verification" of DTFT relationship.

$$x(n) \overset{F}{\longleftrightarrow} X(e^{jw})$$

Consider inverse DTFT

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{X(e^{jw})}_{=\sum_{m=-\infty}^{\infty} x(m)\, e^{-jwm}} e^{jwn}\, dw$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} x(m)\, e^{-jwm}\right) e^{jwn}\, dw$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} x(m) \underbrace{\int_{-\pi}^{\pi} e^{jw(m-n)}\, dw}_{\frac{2\sin\pi(n-m)}{(n-m)}}$$

$$= \sum_{m=-\infty}^{\infty} x(m) \underbrace{\frac{\sin\pi(n-m)}{\pi(n-m)}}_{= \begin{cases} 1 & \text{if } n=m \\ 0 & \text{otherwise} \end{cases}}$$

$$= \sum_{m} x(m)\, \delta(n-m)$$

$$= x(n)$$

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Existense of DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Consider partial sum

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x(n) e^{-j\omega n}$$

Under what conditions on $x(n)$ the limit $\lim_{M\to\infty} X_M(e^{j\omega})$ exist ?

Case I

If $x(n)$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

then DTFT of $x(n)$ exists.

$$\lim_{M\to\infty} X_M(e^{j\omega}) \longrightarrow X(e^{j\omega})$$

for any $\omega \in \mathbb{R}$.

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$$|X(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right|$$

$$\leq \sum_{n=-\infty}^{\infty} \left| x(n) e^{-j\omega n} \right|$$

$$\leq \sum_{n=-\infty}^{\infty} |x(n)| \underbrace{|e^{-j\omega n}|}_{1}$$

$$< \infty$$

$$\lim_{M\to\infty} X_M(e^{j\omega}) \Rightarrow X(e^{j\omega})$$

bounded for any ω.

Case II:

If $x(n)$ has finite energy

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

then

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} |X_M(e^{j\omega}) - X(e^{j\omega})|^2 d\omega = 0$$

DTFT Examples.

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① $x(n) = a^n u(n)\ ;\quad |a| < 1$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

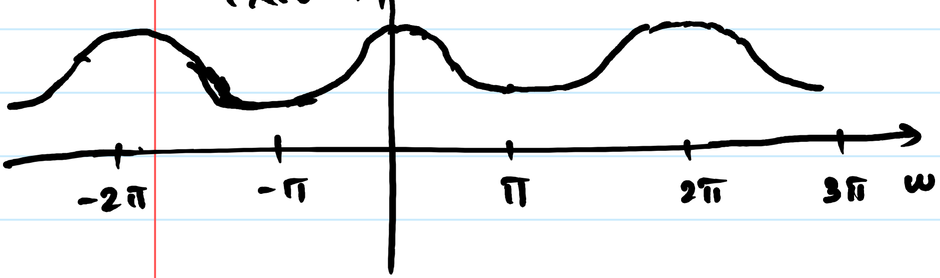
$$= \sum_{n=0}^{\infty} (\underbrace{a e^{-j\omega}}_{r})^n$$

$$= \frac{1}{1 - a e^{-j\omega}} \qquad |r| < 1$$

Plot of spectrum

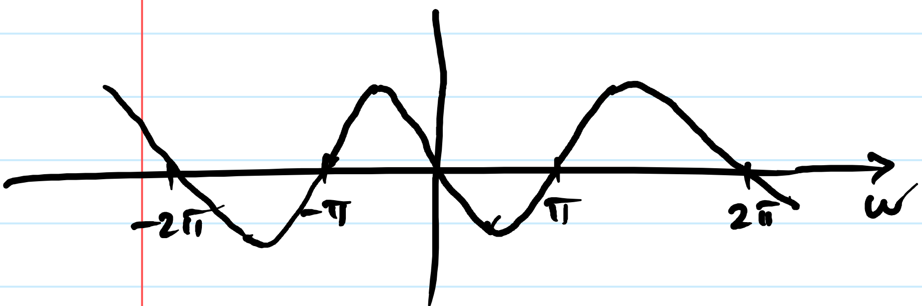
Case 1: $0 < a < 1$

$|X(e^{j\omega})|$ Magnitude spectrum



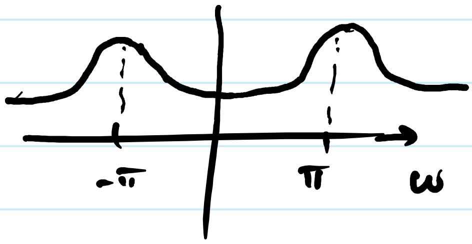
Phase spectrum

$\angle X(e^{j\omega})$



Case $0 > a > -1$

Mag. Spectrum



Phase Spectrum

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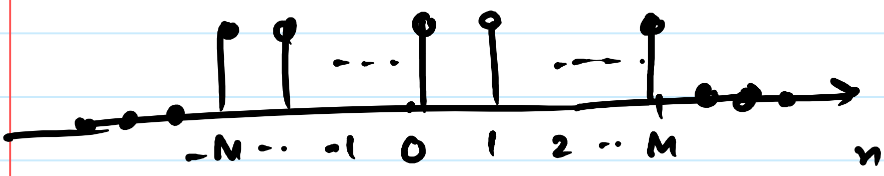
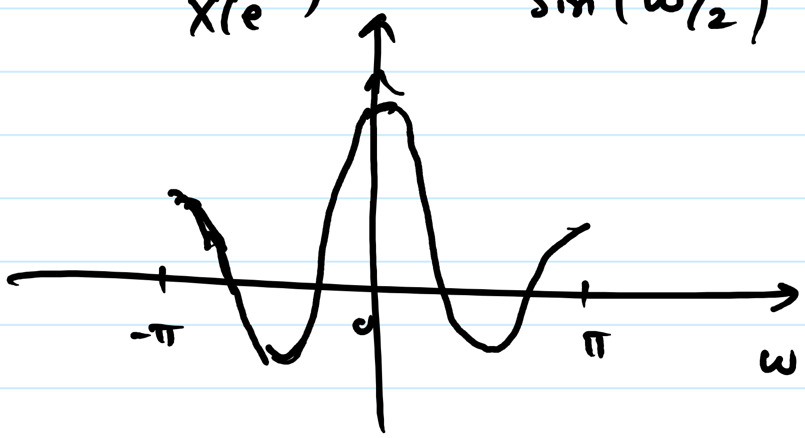
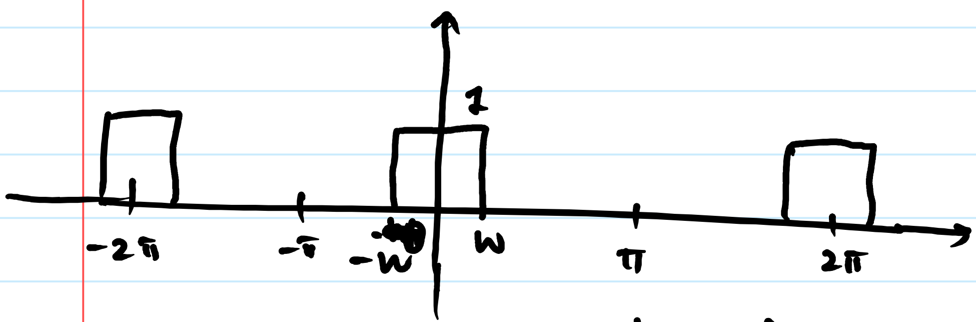
(2) $$x(n) = \begin{cases} 1 & \text{if } |n| \le M \\ 0 & \text{else} \end{cases}$$



$$X(e^{+j\omega}) = \frac{\sin(\omega(M+1/2))}{\sin(\omega/2)}$$



(3) $$X(e^{j\omega}) = \begin{cases} 1 & \text{if } |\omega| \le W \\ 0 & \text{if } \pi \ge |\omega| \ge W \end{cases}$$



$$x(n) = \frac{1}{2\pi} \int_{-W}^{W} 1 \cdot e^{j\omega n} \, d\omega$$

$$= \frac{\sin Wn}{\pi n}$$

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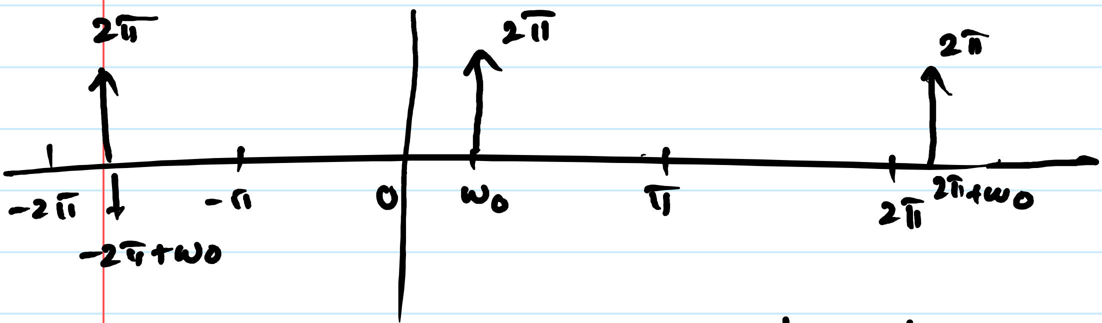
(4) $x(n) = e^{j\omega_0 n}$

has infinite energy

Technically DTFT does not exist

But we find DTFT using impulse functions

~~X(jw)~~ $X(e^{j\omega})$



spectrum has impulse at ω_0

it is periodic

(impulses at $\omega_0 + 2\pi r$, $r \in \mathbb{Z}$)

Area under impulse 2π

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 - 2\pi r)$$

Inverse $\displaystyle\int_{-\pi}^{\pi} \frac{1}{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$

$$= \int_{-\pi}^{\pi} \frac{1}{2\pi} \cdot 2\pi\, \delta(\omega - \omega_0)\, e^{j\omega n}\, d\omega$$

$$= e^{j\omega_0 n}$$

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Symmetry Properties of DTFT

$x(n) \rightarrow$ complex valued Signal/ sequence

$X(e^{j\omega}) \longrightarrow$ Complex valued spectrum

* ———————— *

In general, for complex valued Signal $x(n)$ we have

$$x(n) = x_e(n) + x_o(n)$$

with

→ conjugate

$$x_e(n) = \frac{x(n) + x^*(-n)}{2}$$

$$x_o(n) = \frac{x(n) - x^*(-n)}{2}$$

clearly

$$x_e(n) = x_e^*(-n)$$ (Conjugate-Symmetric sequence)

$$x_o(n) = -x_o^*(-n)$$

Conjugate-anti symmetry

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Similarly any Spectrum $X(e^{jw})$

can be written as

$$X(e^{jw}) = X_e(e^{jw}) + X_o(e^{jw})$$

where

Conjugate Symmetric → $X_e(e^{jw}) = \dfrac{X(e^{jw}) + X^*(e^{-jw})}{2}$

Conjugate anti-symmetric → $X_o(e^{jw}) = \dfrac{X(e^{jw}) - X^*(e^{-jw})}{2}$

Suppose

$$x(n) \xleftrightarrow{F} X(e^{jw})$$

$$x^*(n) \xleftrightarrow{F} X^*(e^{-jw})$$

spectrum of $x^*(n)$ is the

Conjugated and reversed (freq) version

of spectrum of $x(n)$

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Proof:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

Conjugate both sides

$$x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\, e^{-j\omega n}\, d\omega$$

replace ω with $(-\omega)$

$$x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{-j\omega})\, e^{j\omega n}\, d\omega$$

$$x^*(n) \xleftrightarrow{F} X^*(e^{-j\omega})$$

$*$ ——— $*$

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Summary of Symmetry Properties

	Sequence	DTFT
	$x(n)$	$X(e^{jw})$
1	$x^*(n)$	$X^*(e^{-jw})$
2	$x^*(-n)$	$X^*(e^{jw})$
3	$Re\{x(n)\} = \frac{x(n)+x^*(n)}{2}$	$X_e(e^{jw})$
4	$j\,Im\{x(n)\}$	$X_o(e^{jw})$
5	$x_e(n)$	$Re\{X(e^{jw})\}$
6	$x_o(n)$	$j\,Im\{X(e^{jw})\}$
7	$x(n)$ is real $x(n) = x^*(n)$	$X(e^{jw}) = X^*(e^{-jw})$

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When $x(n)$ is real

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

Suppose

$$X(e^{j\omega}) = X_R(e^{j\omega}) + j\,X_I(e^{j\omega})$$

$$X^*(e^{-j\omega}) = X_R(e^{-j\omega}) - j\,X_I(e^{-j\omega})$$

if they are equal

even function $\rightarrow$ $X_R(e^{j\omega}) = X_R(e^{-j\omega})$

odd function $\rightarrow$ $X_I(e^{j\omega}) = -X_I(e^{-j\omega})$

$$|X(e^{j\omega})| = |X(e^{-j\omega})|$$

$$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$$

Example

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real sequence $\rightarrow$ $x(n) = a^n u(n)$; a real, $|a| < 1$

$$X(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

$$X(e^{-j\omega}) = \frac{1}{1 - a e^{j\omega}}$$

$$X^*(e^{-j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

So $X(e^{j\omega}) = X^*(e^{-j\omega})$

Verify the symmetry properties

$\cancel{|X(e^{j\omega})| = |X^*}$

$$|X(e^{j\omega})| = |X(e^{-j\omega})|$$

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General Properties of DTFT

$$x(n) \xleftrightarrow{F} X(e^{j\omega})$$

① Linearity

$$x_1(n) \xleftrightarrow{F} X_1(e^{j\omega})$$

$$x_2(n) \xleftrightarrow{F} X_2(e^{j\omega})$$

$$a x_1(n) + b x_2(n) \xleftrightarrow{F} a X_1(e^{j\omega}) + b X_2(e^{j\omega})$$

② Time Shifting

$$x(n) \xleftrightarrow{F} X(e^{j\omega})$$

$$x(n-N) \xleftrightarrow{F} e^{-j\omega N} X(e^{j\omega})$$

$N \to$ constant

Proof Let $y(n) = x(n-N)$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n-N)\, e^{-j\omega n}$$

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Change of variables $n = m + N$

$m = n - N$

$$Y(e^{j\omega}) = \sum_m x(m)\, e^{-j\omega(m+N)}$$

$$= e^{-j\omega N} \underbrace{\sum_m x(m)\, e^{-j\omega m}}_{X(e^{j\omega})}$$

$$= e^{-j\omega N} X(e^{j\omega})$$

$$x(n-N) \longleftrightarrow e^{-j\omega N} X(e^{j\omega})$$

$\Downarrow$ delay

$\downarrow$ linear phase (in both ω & N)

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(3) Frequency domain shifting

$$x(n) \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$e^{j\omega_0 n} x(n) \longleftrightarrow X(e^{j(\omega-\omega_0)})$$

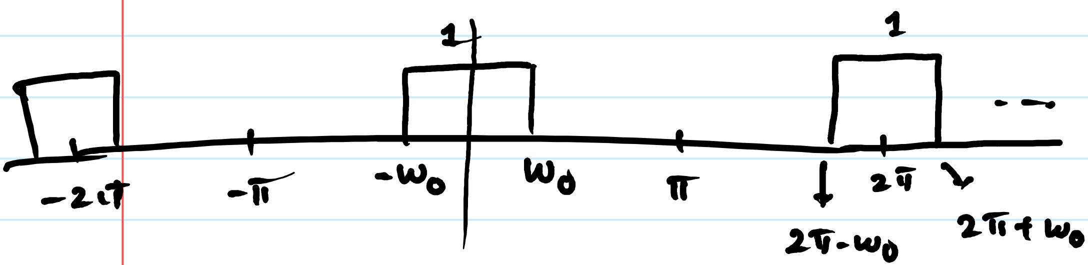
linear phase in time domain

shifting in spectral domain

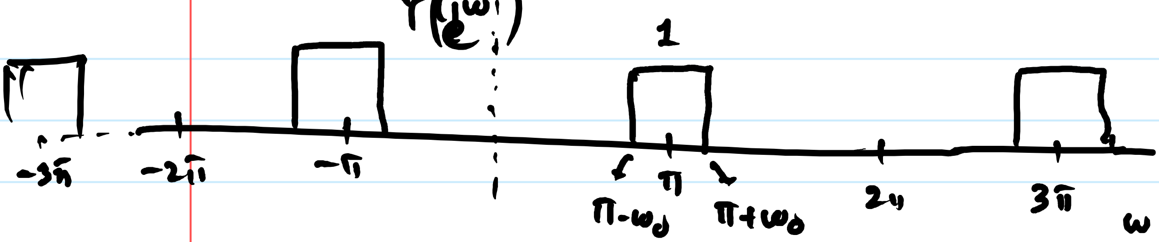
Proof: Do yourself

Example:

$X(e^{j\omega})$



Recall $x(n) = \dfrac{\sin(\omega_0 n)}{\pi n}$



$$Y(e^{j\omega}) = X(e^{j(\omega-\pi)})$$

$$y(n) = e^{j\pi n} x(n)$$

$$= (-1)^n x(n)$$

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4 Differentiation in Spectral domain

$$X(e^{jw}) = \sum_n x(n) e^{-jwn}$$

$$\frac{d}{dw} X(e^{jw}) = \sum_n x(n) \frac{d}{dw} e^{-jwn}$$

$$= \sum_n x(n)(-jn) e^{-jwn}$$

$$j \frac{d}{dw} X(e^{jw}) = \sum_n n\, x(n) e^{-jwn}$$

$$n\, x(n) \xleftrightarrow{F} j \frac{d}{dw} X(e^{jw})$$

x ——————————— x

5 Parseval's theorem

$$x(n) \xleftrightarrow{F} X(e^{jw})$$

Energy of signal

$$= \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{jw})|^2 dw$$

$|X(e^{jw})|^2 \longrightarrow$ Energy Spectral density

(energy in each frequency)

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Proof:

$$\sum_n |x(n)|^2 = \sum_n x(n)\, x^*(n)$$

$$= \sum_n x(n) \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \right]^*$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{j\omega}) \underbrace{\left(\sum_n x(n) e^{-j\omega n} \right)}_{X(e^{j\omega})} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

⑥ Convolution Property.

$$x(n) \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

$$h(n) \overset{\mathcal{F}}{\longleftrightarrow} H(e^{j\omega})$$

Let $y(n) = x(n) * h(n)$

Now $Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$

$$h(n) \overset{\mathcal{F}}{\longleftrightarrow} H(e^{j\omega})$$

$h(n)$ ↓ impulse response of LTI system

$H(e^{j\omega})$ ↓ Frequency response of LTI system

Proof **One idea**

$$y(n) = \sum_{k} x(k)\, h(n-k)$$

$$Y(e^{j\omega}) = \sum_{n} y(n)\, e^{-j\omega n}$$

} after some algebra

$$= X(e^{j\omega})\, H(e^{j\omega})$$

~~Alt~~

Alternative idea

(using eigen functions)

$$e^{j\omega_0 n} \longrightarrow \boxed{\begin{array}{c} \text{LTI} \\ h(n) \end{array}} \longrightarrow H(e^{j\omega_0})\, e^{j\omega_0 n}$$

$$H(e^{j\omega_0}) = \sum_{n} h(n)\, e^{-j\omega_0 n}$$

$$= H(e^{j\omega}) \Big|_{@\, \omega = \omega_0}$$

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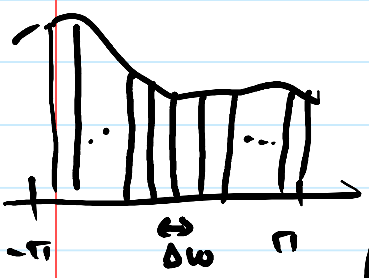
$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$



$$= \frac{1}{2\pi}\lim_{\Delta\omega \to 0}\sum_{k} X(e^{jk\Delta\omega})\, e^{jk\Delta\omega n}\, \Delta\omega$$

$$y(n) = \frac{1}{2\pi}\lim_{\Delta\omega \to 0}\sum_{k} X(e^{jk\Delta\omega})\, H(e^{jk\Delta\omega})\, e^{jk\Delta\omega n}\, \Delta\omega$$

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, H(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$\downarrow$

looks like an IDFT

$$Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$$

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⑦ Windowing property

$$x(n) \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$w(n) \xleftrightarrow{\mathcal{F}} W(e^{j\omega})$$

$$x(n)\,w(n) \xleftrightarrow{F} \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})\,W(e^{j(\omega-\theta)})\,d\theta}_{\text{(periodic) Convolution of Spectrums}}$$

x ———————— x

Application - LCCDE

Finding impulse response of an LTI system corresponding to an LCCDE (which is initially at rest)

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Example:

$$y(n) - \frac{1}{2}y(n-1) = x(n) - \frac{x(n-1)}{4}$$

Say input $x(n) = \delta(n)$

output $y(n) = h(n)$

$$h(n) - \frac{1}{2}h(n-1) = \delta(n) - \frac{1}{4}\delta(n-1)$$

Taking DTFT on both sides

$$H(e^{j\omega}) - \frac{1}{2}e^{-j\omega}H(e^{j\omega})$$

$$= \underbrace{1}_{\delta(n) \overset{F}{\longleftrightarrow} 1} - \frac{1}{4}e^{-j\omega}$$

$$H(e^{j\omega})\left(1 - \frac{1}{2}e^{-j\omega}\right) = 1 - \frac{1}{4}e^{-j\omega}$$

$$H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} - \frac{\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

$$h(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)^{n-1}u(n-1)$$