Dft

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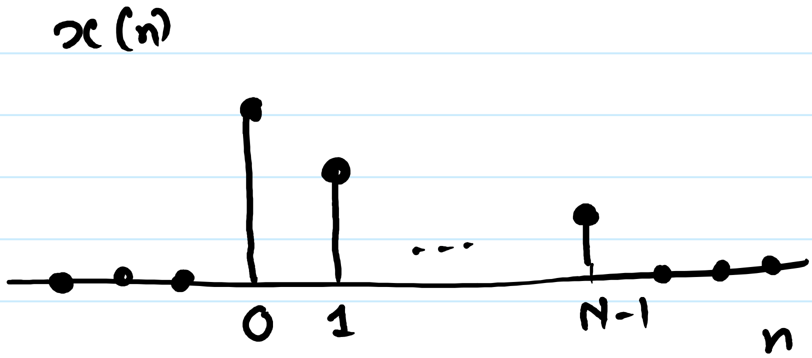
Discrete Fourier Transform (DFT)

- Most practically used tool in Fourier analysis

Consider a finite length sequence

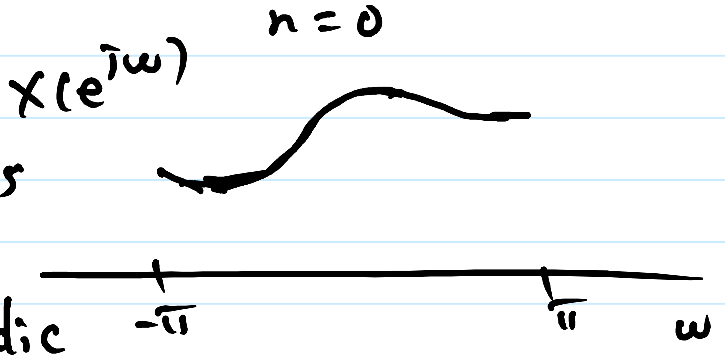
$x(n) = 0$ for $n < 0$,
$n \geq N$



Its spectrum

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n}$$

$\omega \rightarrow$ continuous variable



spectrum periodic

Question :

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Can we get <u>full</u> information of $x(n)$ using few <u>samples</u> of the spectrum ?

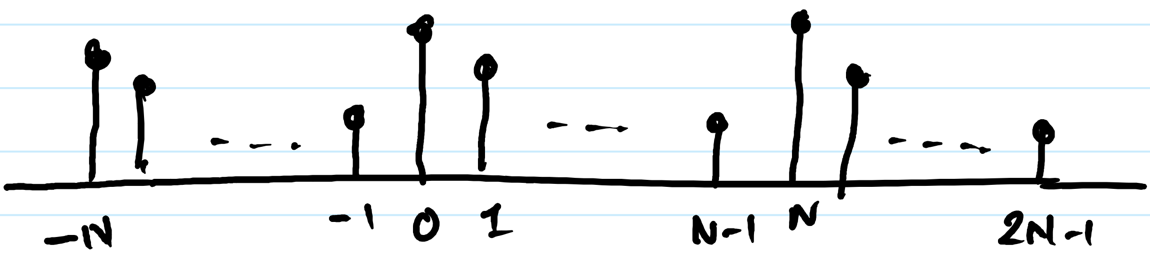
Consider a related problem :

Consider a periodic signal $\tilde{x}(n)$ with period N

such that

$$\tilde{x}(n) = \begin{cases} x(n) & 0 \le n \le N-1 \end{cases}$$

$$\tilde{x}(n+N) = \tilde{x}(n)$$



$\tilde{x}(n)$ is $x(n)$ repeated periodically

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$\tilde{x}(n)$ is identical to $x(n)$

for $n = 0$ to $N-1$

$\tilde{x}(n)$ and $x(n)$ have same information

knowing $x(n)$ we can get $\tilde{x}(n)$

and vice Versa.

Questions:

- What is the spectrum of $\tilde{x}(n)$?
- How is spectrum of $\tilde{x}(n)$ is related to spectrum of $x(n)$?

We introduce Discrete Fourier Series (DFS)

to answer these questions!

Discrete Fourier Series

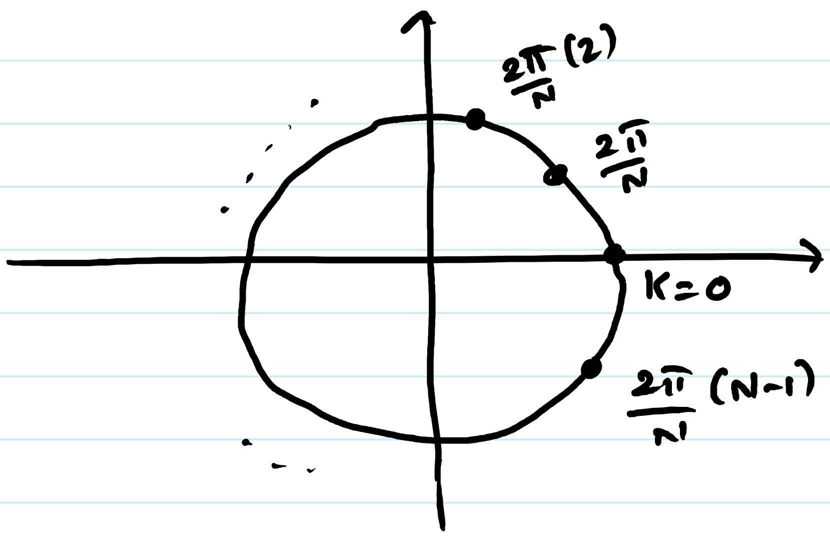
$\tilde{x}(n)$ is periodic

Time Period N (samples)

fundamental frequency $\frac{2\pi}{N}$ radians/sample

$\tilde{x}(n)$ can be written as weighted sum of Complex exponentials of fundamental freq. & its harmonics

k^{th} harmonic : $e^{j\frac{2\pi}{N}kn}$



Note that there are only N <u>distinct</u> harmonics

harmonics

$$\left\{ e^{j\frac{2\pi}{N}kn} \;;\; k = 0, 1, .., N-1 \right\}$$

are the N distinct harmonics

$K = N$ is same as $k = 0$

$K = N+1$ is same as $k = 1$

$K = -1$ is same as $k = N-1$

and so on

Hence

$$\tilde{x}(n) = \sum_{K=0}^{N-1} \frac{a_k}{N} e^{j\frac{2\pi}{N}kn}$$

$\frac{a_k}{N} \rightarrow$ Fourier coefficient of k^{th} harmonic

Note $\frac{1}{N}$ is there for <u>convenience</u>.

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Now, a_k is computed as

$$a_k = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\frac{2\pi}{N}kn}$$

<u>Proof:</u>

$$\sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \left(\sum_{l=0}^{N-1} \frac{a_l}{N} e^{j\frac{2\pi}{N}ln} \right) \times e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{l=0}^{N-1} a_l \underbrace{\frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(l-k)n}}_{= \begin{cases} 1 & \text{if } k=l \\ 0 & \text{else} \end{cases}}$$

$$= a_k$$

∎

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Summarizing

DFS

$$a_k = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\frac{2\pi}{N}kn}$$

$k = 0, 1, \cdots, N-1$

$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N} a_k\, e^{j\frac{2\pi}{N}kn}$$

$n \in \mathbb{Z}$

$-\infty < n < \infty$

Spectrum of $\tilde{x}(n)$ has only N frequencies (harmonics)

$\tilde{X}(e^{j\omega})$ be spectrum of $\tilde{x}(n)$

$$\tilde{X}(e^{j\omega}) = \frac{1}{N} \sum_{k=0}^{N-1} a_k\, 2\pi\, \delta\!\left(\omega - \frac{2\pi}{N}k\right)$$

$0 \leq \omega \leq 2\pi$

and this gets repeated periodically.

Since $x(n)$ Coincides with

$\tilde{x}(n)$ in the interval
$n = 0, 1, \ldots, N-1$

we have the following

DFT $\rightarrow$ $$X[k] = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}nk}$$

$$0 \leq k \leq N-1$$

inverse DFT $\rightarrow$ $$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}$$

$$0 \leq n \leq N-1$$

In the inverse DFT, we restrict our time index n from 0 to $N-1$

If we extend the time index beyond these values, we get periodic signal $\tilde{x}(n)$

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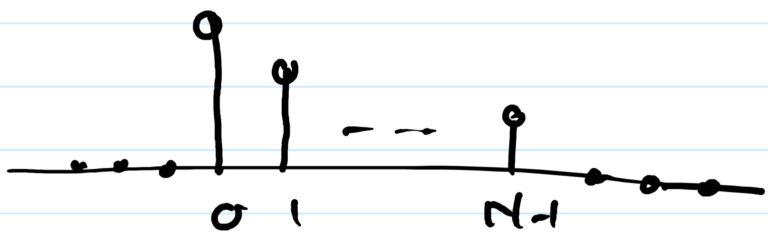
$$\{ X[k] \, , \; 0 \le k \le N-1 \}$$

are referred as DFT coefficients

How are the DFT coefficients related to spectrum of $x(n)$?



Finite duration $x(n)$ is 0 for $\begin{cases} n \geq N \\ n < 0 \end{cases}$

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) \, e^{-j\omega n}$$

Note : $$X(k) = \sum_{n=0}^{N-1} x(n) \, e^{-j\frac{2\pi}{N}kn}$$

Hence $$X[k] = X(e^{j\omega}) \Big|_{\omega = \frac{2\pi k}{N}}$$

$$k = 0, 1, \dots, N-1$$

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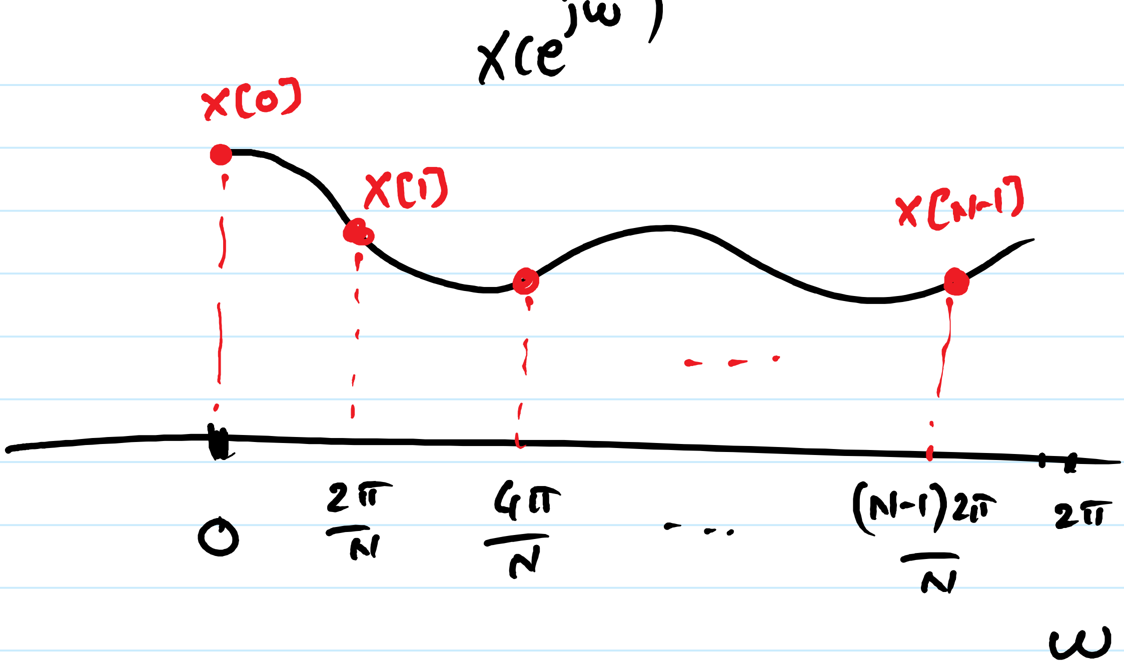

For a finite duration signal (N length)

DFT coefficients are identical to the N equally spaced samples in the spectrum

DFT is equivalent to sampling the DTFT (equi spaced)

N length time domain signal

$\Leftrightarrow$ (equivalent) N equi spaced samples in spectrum

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Matrix Representation of DFT

Consider N-length signal

$$x(n) = \{x(0), \dots, x(N-1)\}$$

$$\uparrow$$

Consider the N - DFT coefficients

$$X[k] = \{X[0], X[1], \dots, X[N-1]\}$$

Let us define vectors of Signal & DFT coeff.

$$\underline{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \qquad \underline{X} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$N \times 1$ $\qquad$ $N \times 1$

Recall

$$X[k] = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}nk}$$

$$X[k] = \begin{bmatrix} e^{-j\frac{2\pi}{N}k(0)} & e^{-j\frac{2\pi}{N}k(1)} & \cdots & e^{-j\frac{2\pi}{N}k(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

stacking $X[k]$, $k = 0, \ldots, N-1$

DFT coefficients ↓

Signal Samples ↓

$$\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X(N-1] \end{bmatrix} = \underbrace{\begin{bmatrix} e^{-j\frac{2\pi}{N}(0)(0)} & \cdots & e^{-j\frac{2\pi}{N}(0)(N-1)} \\ e^{-j\frac{2\pi}{N}(1)(0)} & \cdots & e^{-j\frac{2\pi}{N}(1)(N-1)} \\ \vdots & & \\ e^{-j\frac{2\pi}{N}(N-1)(0)} & \cdots & e^{-j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix}}_{F_N} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

N point DFT $\underline{X} = F_N \underline{x}$

$F_N \rightarrow$ N point DFT matrix

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Easy to verify that

Inverse matrix $F_N^{-1} = \frac{1}{N} F_N^{*}$

$* \rightarrow$ conjugate and transpose

$$F_N^{*} = \begin{bmatrix} e^{j\frac{2\pi}{N}(0)(0)} & \cdots & e^{j\frac{2\pi}{N}(N-1)(0)} \\ \vdots & & \vdots \\ e^{j\frac{2\pi}{N}(0)(N-1)} & & e^{j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix}$$

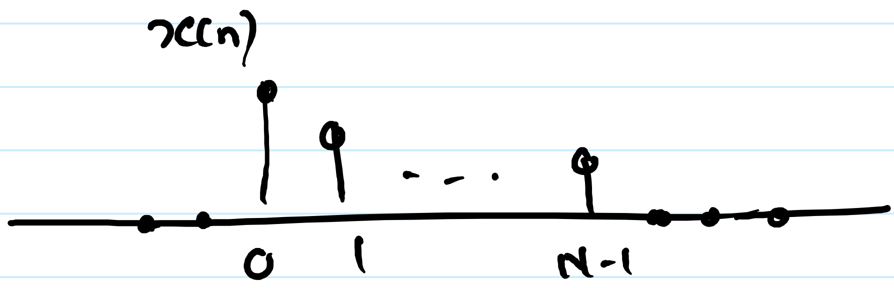
inverse DFT (N point) $\underline{x} = \frac{1}{N} F_N^{*} \underline{X}$

From above matrix representation, we get

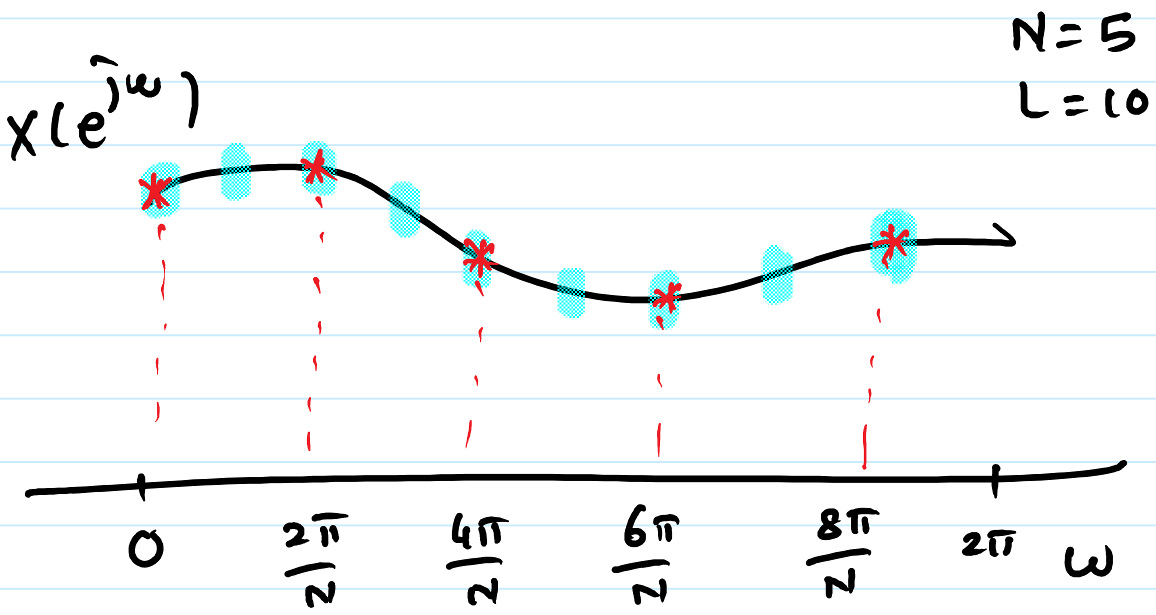
$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

Concept of zero-padding

Suppose we have N length signal



Say we compute N pt DFT of $x(n)$



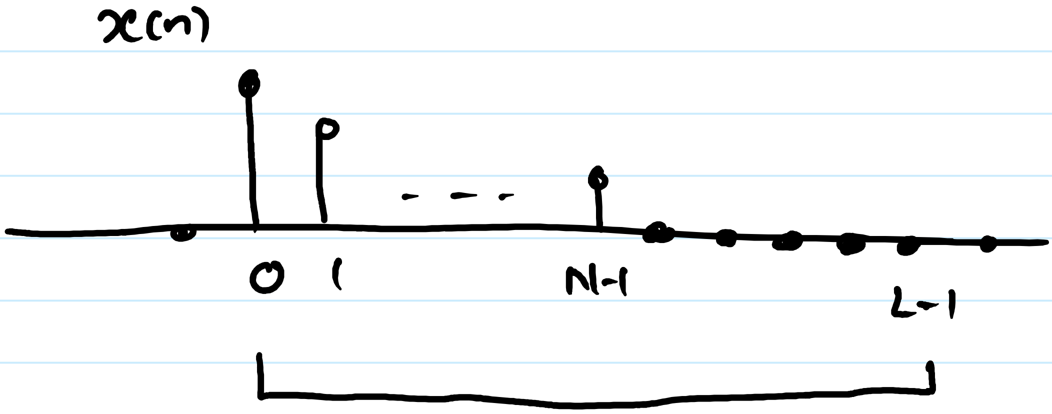
* → coeff of N pt DFT

▮ → Coeff required

Suppose we want L samples of spectrum

(L > N)

This can be done by
taking L pt DFT



Compute L point DFT
(by doing zero padding)

Let
$$\underline{x}_L = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \\ \left.\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\} L-N \end{bmatrix} \text{zeros}$$

$F_L \rightarrow$ L point DFT matrix

entries $\left\{ e^{-j\frac{2\pi}{L}kn} \right\}$ $\quad 0 \le k \le L-1$, $\; 0 \le n \le N-1$

$$\underline{X}_L = F_L \, \underline{x}_L$$

L samples of spectrum

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Note that

L need not be a multiple of N

By zero padding a finite duration signal and taking a larger point DFT improves the resolution of sampling in the spectrum.

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Properties of DFT

① Linearity

$x_1(n) \Rightarrow$ length N_1 signal

$\{x_1(0), x_1(1), \ldots, x_1(N_1-1)\}$
$\uparrow$

$x_2(n) \Rightarrow$ length N_2 signal

$\{x_2(0), x_2(1), \ldots, x_2(N_2-1)\}$

Let $y(n) = a\,x_1(n) + b\,x_2(n)$

$y(n) \Rightarrow$ length $N_3 = \max(N_1, N_2)$

Choose $N \geq N_3$

Suppose

$$x_1(n) \xleftrightarrow[\text{DFT}]{N\text{ pt}} X_1[k]$$

$$x_2(n) \xleftrightarrow[\text{DFT}]{N\text{ pt}} X_2[k]$$

$$y(n) \xleftrightarrow[\text{DFT}]{N\text{ pt}} Y[k]$$

Now, we have

$$Y[k] = a X_1[k] + b X_2[k]$$

$$k = 0, 1, \ldots, N-1$$

When we talk about DFT, the size of DFT is also need to be mentioned

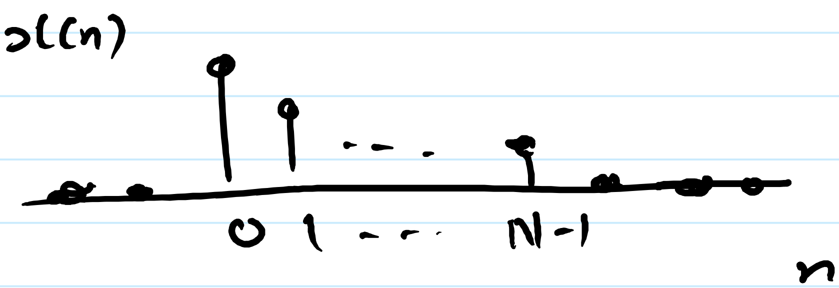
o ——— x

Property 2

Time Shifting (circular)

$x(n) \rightarrow$ length N signal

$n = 0$ to $N-1$



Let

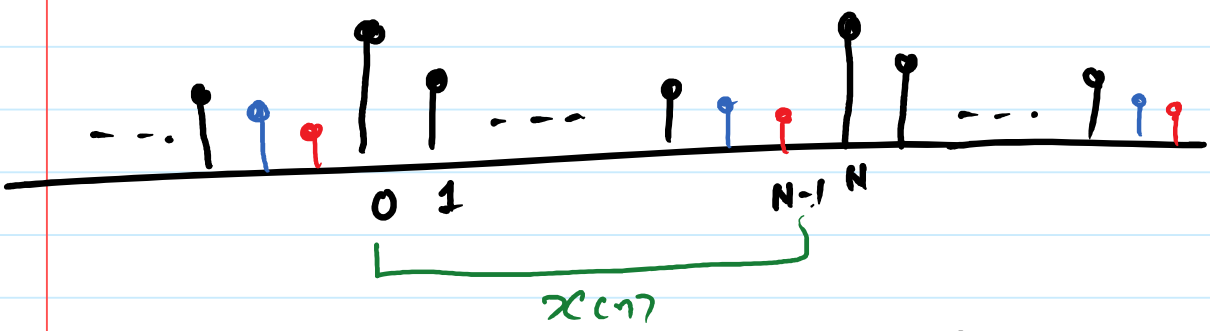
$$x(n) \xleftrightarrow[\text{DFT}]{N\text{ pt}} X[k]$$

We have

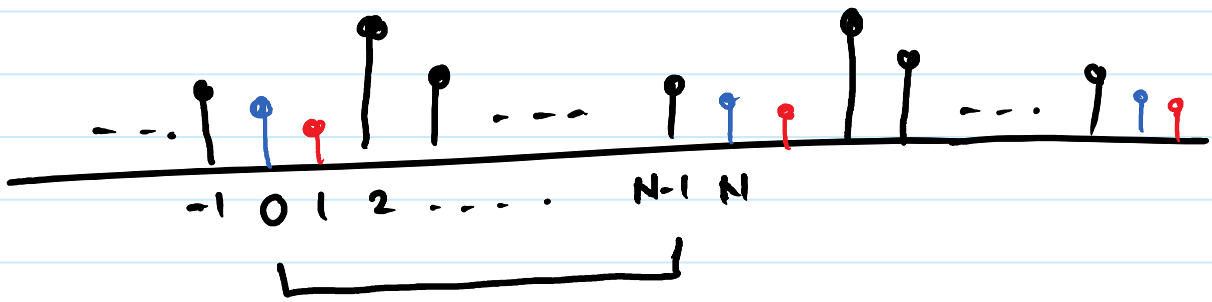
$$x(n) = \sum_{k=0}^{N-1} \frac{X[k]}{N} e^{j\frac{2\pi}{N}n}$$

RHS is inherently periodic

$\tilde{x}(n)$



Consider time shift $\tilde{x}(n-2)$



Time shifting $\tilde{x}(n)$ has a circular shift on $x(n)$

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Circular shift can be represented using mod_N indexing

$\text{mod}_N \Rightarrow$ modulo N operation

$a \bmod N \Rightarrow$ remainder when you divide a by N

$\Rightarrow$ Note remainder should be from the set $\{0, 1, \ldots, N-1\}$

$1 \bmod N = 1$

$N \bmod N = 0$

$N+1 \bmod N = 1$

$-1 \bmod N = N-1$

$-2 \bmod N = N-2$

and so on.

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Let $y(n)$ be circular shift of $x(n)$ by 'm' units (samples)

Now $y(n) = x((n-m) \bmod N)$

$n = 0, 1, \dots, N-1$

Let $x(n) \underset{DFT}{\overset{N}{\longleftrightarrow}} X[k]$

$y(n) \underset{DFT}{\overset{N}{\longleftrightarrow}} Y[k]$

We have

$$Y[k] = e^{-j\frac{2\pi}{N}mk} X[k]$$

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Symmetry Properties

(a) $x(n) \xleftrightarrow[DFT]{N} X[k]$

$k = 0, 1, \ldots, N-1$

then $x^*(n) \xleftrightarrow[DFT]{N} X^*(-k \bmod N)$

$k = 0, 1, \ldots, N-1$

In otherwords

ordered set

$$x(n) \xrightarrow{DFT} \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$x^*(n) \xrightarrow{DFT} \begin{bmatrix} X^*[0] \\ X^*(N-1) \\ X^*(N-2) \\ \vdots \\ X^*(1) \end{bmatrix}$$

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$$x(n) \xleftrightarrow{DFT} X[k]$$

(b) $x(-n \bmod N)$

$$\xleftrightarrow[DFT]{} X^*[k]$$

$$x(-n \bmod N) = \{x(0), x(N-1), x(N-2), \ldots, x(1)\}$$

(c) If $x(n)$ is real valued

$$x(n) = x^*(n)$$

then

$$X(k) = X^*(-k \bmod N)$$

that is

$$X[k] = X^*[N-k]$$

$$k = 1, 2, \ldots, N-1$$

$$X[0] = X^*[0]$$

(d) If $x(n)$ is even symmetry

$$x(n) = x(N-n)$$

$n = 1, 2, \ldots, N-1$

then $X[k] = X^*[k]$

That is, DFT coefficients are real.

Circular Convolution Property

$x_1(n) \rightarrow$ length N signal

$x_2(n) \rightarrow$ length N signal

Let $x_1(n) \xleftrightarrow[\text{DFT}]{N\text{ pt}} X_1[k]$

$x_2(n) \xleftrightarrow[\text{DFT}]{N\text{ pt}} X_2[k]$

Let $Y[k] = X_1[k] X_2[k]$

Product of DFT coef.

& $y(n)$ be time domain signal

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Corresponding to DFT coeff $Y[k]$

$$y(n) \underset{DFT}{\overset{N}{\longleftrightarrow}} Y[k]$$

How is $y(n)$ related to $x_1(n)$, $x_2(n)$?

Answer:

$$y(n) = \sum_{k=0}^{N-1} x_1(k)\, \underbrace{x_2(n-k \bmod N)}_{\text{circular shift}}$$

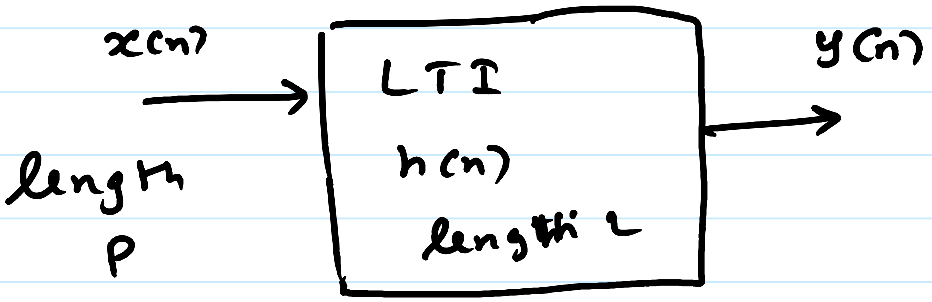
$n = 0, \dots, N-1$

This is called circular convolution

$$y(n) = \underbrace{x_1(n) \; (N) \; x_2(n)}_{}$$

Notation (length N circular convolution)

How to implement linear convolution using DFT ?



$$x(n) = \{\underset{\uparrow}{x(0)}, x(1), \dots, x(P-1)\}$$

$$h(n) = \{\underset{\uparrow}{h(0)}, h(1), \dots, h(L-1)\}$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

↓ linear convolution

$$y(n) = \sum_{k=0}^{P-1} x(k)\, h(n-k)$$

$$n = 0, 1, \dots, L+P-1$$

$y(n) \rightarrow$ length $L+P-1$

Choose $\boxed{N \geq L+P-1}$

Now,

Let $x(n) \xleftrightarrow[\text{DFT}]{N \text{ pt}} X[k]$

zero pad

$h(n) \xleftrightarrow[\text{DFT}]{N \text{ pt}} H[k]$

zero pad

Compute $V[k] = X[k]\,H[k]$

$k = 0, 1, \ldots, N-1$

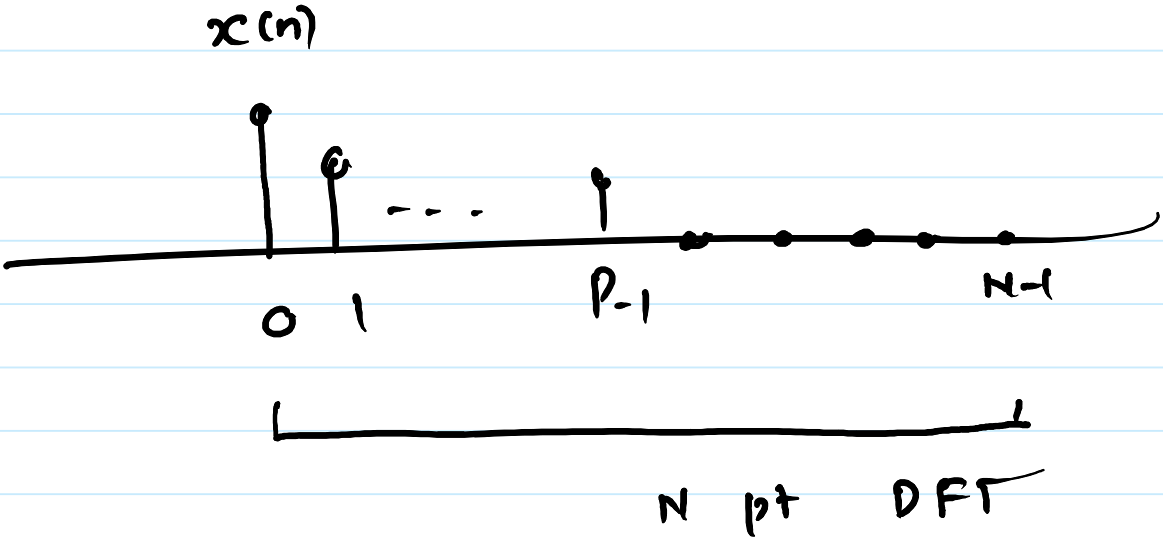
Let $v(n)$ be inverse DFT of $V[k]$

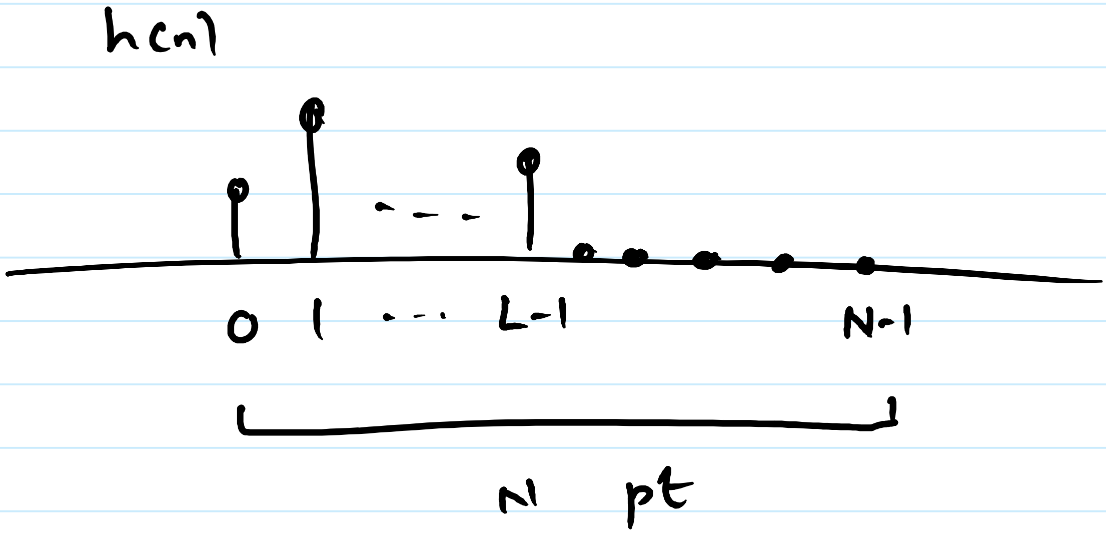
$v(n) \xleftrightarrow[\text{DFT}]{N \text{ pt}} V[k]$

Claim : $v(n) = y(n)$

with sufficient zero padding,

linear convolution can be obtained thru circular convolution





Due to presence of zeros, circular shifting becomes equivalent to linear time shifting

Proof:

Easy to interpret in DFT domain

We know

$x(n) \underset{DFT}{\overset{N}{\longleftrightarrow}} X(k)$

$h(n) \underset{DFT}{\overset{N}{\longleftrightarrow}} H(k)$

$$y(n) = x(n) * h(n)$$

$$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$

$$y(n) \xrightarrow{N \text{ pt DFT}} Y(k)$$

$$N \geq L + P - 1$$
(length of $y(n)$)

$$Y(k) = Y(e^{j\omega}) \Big|_{\omega = \frac{2\pi}{N}k}$$

$$= X(e^{j\omega}) H(e^{j\omega}) \Big|_{\omega = \frac{2\pi}{N}k}$$

$$= X[k] H[k]$$

$$Y(k) = V[k] \quad , k = 0, 1, \cdots, N-1$$

So $y(n) = v(n)$, $n = 0, 1, \cdots, N-1$

$$x \longrightarrow \infty$$