ComExp

17 January 2018 08:51

Complex Exponentials

defined for both CT & DT cases

Continuous time Case

most general form.

$$x(t) = A e^{st}, \quad -\infty < t < \infty$$

$A, s \in \mathbb{C}$
complex number

Complex Amplitude : A

Complex frequency : s

Let $$A = |A| e^{j\phi}$$

$$s = \sigma_0 + j\Omega_0$$

Now $$x(t) = |A| e^{\sigma_0 t} e^{j(\Omega_0 t + \phi)}$$

$$Re\{x(t)\} = |A| e^{\sigma_0 t} \cos(\Omega_0 t + \phi)$$

$$Im\{x(t)\} = |A| e^{\sigma_0 t} \sin(\Omega_0 t + \phi)$$

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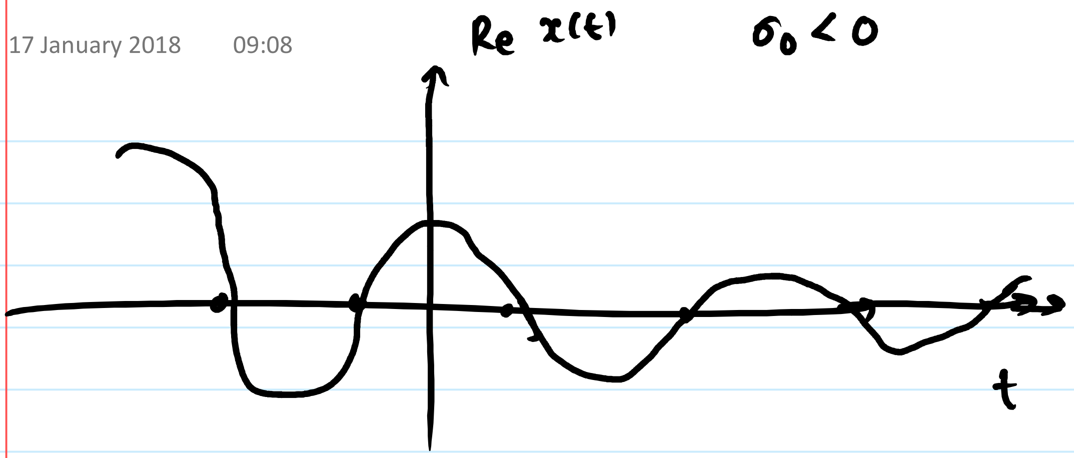


Important Special case:

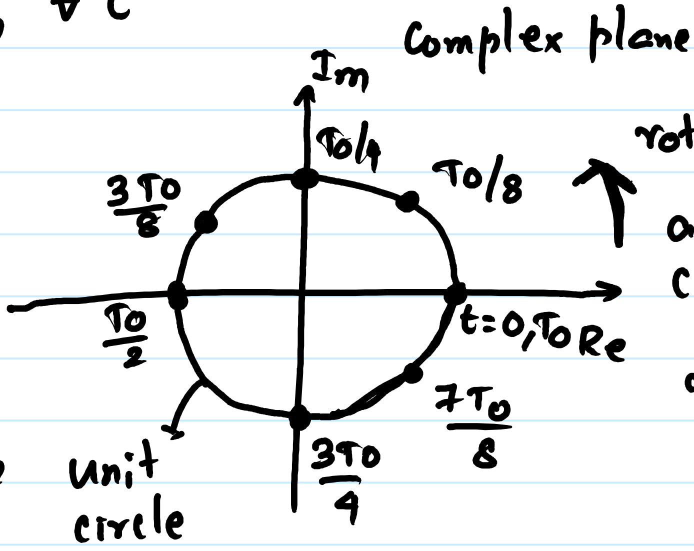
$A = 1$

$s = j\Omega_0 \qquad (\sigma_0 = 0)$

$x(t) = e^{j\Omega_0 t} \qquad$ (Assume $\Omega_0 > 0$)

Define $T_0 = \dfrac{2\pi}{\Omega_0}$

Note $|x(t)| = 1, \ \forall t$



rotates in anti-clockwise direction

Let $t = 0 \ ; \ x(t) = 1$

$t = \dfrac{T_0}{8} \ ; \ x(t) = e^{j\frac{2\pi}{8}}$

$t = T_0/4 \ ; \ x(t) = e^{j\pi/2}$

$t = T_0 \ ; \ x(t) = 1$

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$$x(t+T_0) = e^{j\Omega_0\left(t+\frac{2\pi}{\Omega_0}\right)}$$

$$= e^{j\Omega_0 t} \cdot e^{j2\pi}$$

$$= e^{j\Omega_0 t}$$

$$= x(t)$$

$x(t) = e^{j\Omega_0 t}$ is (always) periodic with fundemental period

$$T_0 = \frac{2\pi}{\Omega_0}$$

- It completes one cycle around unit circle in T_0 seconds (anti-clockwise) direction
- $\Omega_0 \rightarrow$ Angular frequency unit is radians / second
- As Ω_0 increases, the oscillations around unitcircle get faster

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Consider $y(t) = e^{-j\Omega_0 t}$

$\Omega_0 > 0$

$y(t)$ completes one cycle around unit circle in T

&

$2\pi/\Omega_0$ seconds

in the clockwise direction

$e^{j\Omega_0 t}$, $e^{-j\Omega_0 t}$

+ve frequency -ve frequency

They are distinct

Discrete-time Complex Exponentials

General form

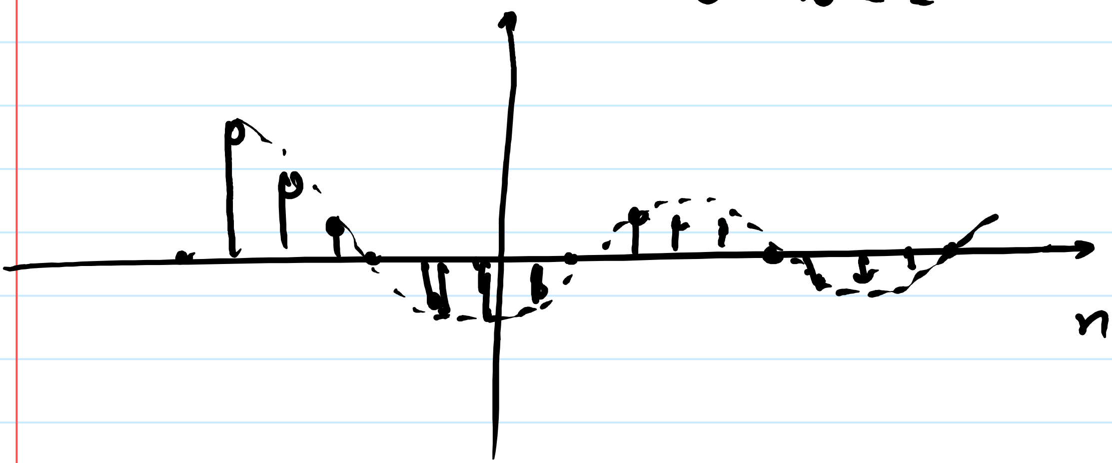
$$x[n] = A\,(z_0)^n, \quad n \in \mathbb{Z}$$

$$A = |A|\,e^{j\phi}$$

$$z_0 = r_0\,e^{j\omega_0} \rightarrow \text{polar form}$$

$$x[n] = |A|\,r_0^n\,e^{j(\omega_0 n + \phi)}, \quad n \in \mathbb{Z}$$

$Re\{x(n)\}$ $\qquad 0 < r_0 < 1$



Special case : $A = 1$

$$z_0 = e^{j\omega_0}$$

$$x(n) = e^{j\omega_0 n}$$

Questions :

1. Is $x(n)$ always periodic ?
2. Does the speed of ~~...~~ rotation increase with ω_0 ?

Periodicity

Suppose $e^{j\omega_0 n}$ is periodic with period N

Then

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}$$

$$\Rightarrow \quad \omega_0 N = 2\pi k$$

where $k \in \mathbb{Z}$

$$\omega_0 = \frac{2\pi k}{N}$$

, k & N are integers

Condition for periodic $e^{j\omega_0 n}$

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② Maximum frequency of rotation

$$x(n) = e^{j\omega_0 n}, \quad n \in \mathbb{Z}$$

$$= \cos(\omega_0 n) + j\sin(\omega_0 n)$$

Consider $\omega_0 + 2\pi$

(angular frequency)

radians/sample

$$e^{j(\omega_0+2\pi)n} = e^{j\omega_0 n} \cdot \underbrace{e^{j2\pi n}}_{1}$$

$$= e^{j\omega_0 n}$$

ω_0 & $\omega_0 + 2\pi$ give same/identical signals

So, discrete time complex exponentials have distinct frequencies in the interval $[0, 2\pi]$ or equivalently $[-\pi, \pi]$

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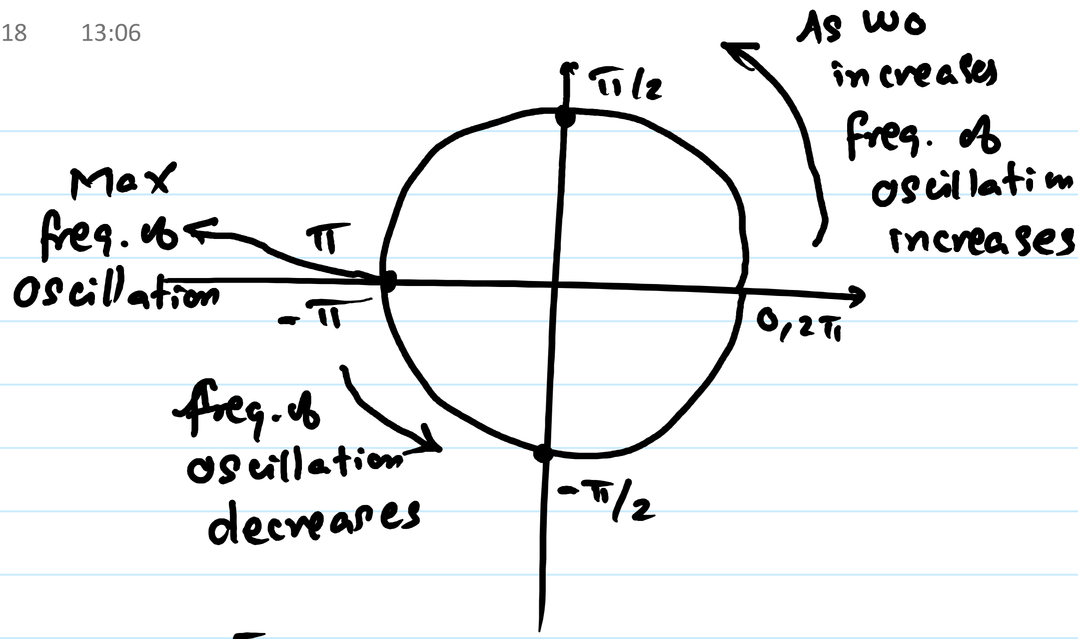

$\omega_0 = 0, 2\pi$

$x(n) = 1$, for all n

(dc sequence)

$\omega_0 = \pi$ or $-\pi$

$x(n) = (-1)^n$

Max. freq. of oscillation