Further Constructive Results on Interconnection and Damping Assignment Control of Mechanical Systems: The Acrobot Example

Arun D. Mahindrakar, Alessandro Astolfi, Romeo Ortega and Giuseppe Viola

Abstract—Interconnection and damping assignment passivity-based control is a controller design methodology that achieves (asymptotic) stabilization of mechanical systems with a Hamiltonian structure and a desired energy function that qualifies as Lyapunov function for the desired equilibrium. The assignable energy functions are characterized by a set of partial differential equations that must be solved to determine the control law. A class of underactuation degree one systems for which the partial differential equations can be explicitly solved—making the procedure truly constructive—was recently reported by the authors. In this brief note, largely motivated by the interesting Acrobot example, we pursue this investigation for two degrees-of-freedom systems where a constant inertia matrix can be assigned. We concentrate then our attention on potential energy shaping and give conditions under which an explicit solution of the associated partial differential equation can be obtained. Using these results we show that it is possible to swing-up the Acrobot from some configuration positions in the lower half plane, provided some conditions on the robot parameters are satisfied.

Notation: For functions $f(q), f : \mathbb{R}^n \to \mathbb{R}$, we denote their gradient and Hessian as $\nabla_q f \triangleq \frac{\partial f}{\partial q}$, $\nabla^2_q f \triangleq \frac{\partial^2 f}{\partial q \partial q}$, respectively, and $\nabla_j f \triangleq \frac{\partial f}{\partial q_j}$, $\nabla_{ij} f \triangleq \frac{\partial^2 f}{\partial q_i \partial q_j}$—when clear from the context the subindex in $\nabla$ will be omitted. For functions of one variable, $(\cdot)'$ will be used to denote its derivative. All vectors are column vectors, even the gradient of a scalar function.

I. INTRODUCTION AND BACKGROUND

In [9] we introduced the interconnection and damping assignment passivity–based control (IDA–PBC) approach to regulate the position of underactuated mechanical systems of the form

$$
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
0 & I_n \\
-I_n & 0
\end{bmatrix}
\begin{bmatrix}
\nabla_q E \\
\nabla_p E
\end{bmatrix} +
\begin{bmatrix}
0 \\
G(q)
\end{bmatrix} u,
$$

(1)

where $q \in \mathbb{R}^n$, $p \in \mathbb{R}^n$ are the generalized positions and momenta, respectively, $u \in \mathbb{R}^m$ and $G \in \mathbb{R}^{n \times m}$ with rank $G = m < n$.

$$
E(q,p) = \frac{1}{2} p^T M^{-1}(q) p + V(q)
$$

(2)

is the total energy with $M = M^T > 0$ the inertia matrix, and $V$ the potential energy. The main result of [9] is the proof that for all matrices $M_d(q) = M^+_d(q) \in \mathbb{R}^{n \times n}$ and functions $V_d(q)$ that satisfy the PDEs

$$
\begin{align*}
0 &= G^+ \{ \nabla_q (p^T M^{-1} p) - M_d M^{-1} \nabla_q (p^T M^{-1} p) \\
&+ 2 J_2 M^{-1} p \} \\
0 &= G^+ \{ \nabla V - M_d M^{-1} \nabla V_d \},
\end{align*}
$$

(3)

for some $J_2(q,p) = -J_2(q,p) \in \mathbb{R}^{n \times n}$ and a full rank left annihilator $G^+(q) \in \mathbb{R}^{(n-m) \times n}$ of $G$, i.e., $G^+ G = 0$ and rank$(G^+) = n - m$, the system (1) in closed–loop with the IDA–PBC

$$
u = (G^T G)^{-1} G^T (\nabla_q H - M_d q M^{-1} \nabla_q V_d + J_2 M^{-1} p),
$$

(5)

takes the Hamiltonian form

$$
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
0 & M^{-1} M_d \\
-M_d M^{-1} & J_2
\end{bmatrix}
\begin{bmatrix}
\nabla_q H_d \\
\nabla_p H_d
\end{bmatrix},
$$

(6)

where the new total energy function is

$$
H_d(q,p) = \frac{1}{2} p^T M^{-1} p + V_d(q).
$$

Further, if $M_d$ is positive definite in a neighborhood of $q^* \in \mathbb{R}^n$ and

$$q^* = \arg \min V_d(q),
$$

(8)

then $(q^*,0)$ is a stable equilibrium point of (6) with Lyapunov function $H_d$.

II. PROBLEM FORMULATION

Clearly, the success of the IDA–PBC method relies on the possibility of solving the PDEs (3), (4) that identify the energy functions that can be assigned to the closed–loop. In [5] the authors give a series of conditions on the system and the assignable inertia matrices such that the PDEs can be solved. Also, techniques to solve the PDEs have been reported in [2], [4] and some geometric aspects of the equations are investigated in [8]. In [7] it is shown that the kinetic energy PDE (3) reduces to an ordinary differential

1The authors study the PDEs appearing in the Controlled Lagrangian method, but as shown in [4], [6] the PDEs are similar.
equation (ODE) if the system is of underactuation degree one, that is, if the difference between the number of degrees of freedom and the number of control actions is one—see also [3] for a detailed study of this case. More recently, in [1], we furthermore proved that if the inertia matrix and the force induced by the potential energy (on the unactuated coordinate) are independent of the unactuated coordinate then the PDEs can be explicitly solved.

In this brief note we pursue the investigation of [1] with the aim of relaxing the aforementioned assumption on the potential energy, hence we keep the assumptions A1.
The system has underactuation degree one.

A2. The inertia matrix $M$ depends only on the actuated coordinates.

Further, motivated by the interesting Acrobot example [10], we make the following assumption:

A3. The system has two degrees-of-freedom and, without loss of generality, take $G = [0 \ 1]^T$.

Assumption A3 is critical for our developments. Assumptions A1 and A2, ensures the term $G^\top \nabla_q (p^\top M^{-1} p)$ in the PDE (3) is zero. In this case (3) can be solved with a constant $M_d$, taking $J_2 = 0$. This allows us to concentrate our attention on potential energy shaping and the PDE to be solved takes the form

$$
\gamma^\top (q_2) \nabla V_d = \nabla_1 V, \quad (9)
$$

where we have used $G^\top = [1 \ 0]$, defined

$$
\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \triangleq M^{-1} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}, \quad (10)
$$

and taken the constant matrix $M_d$ as

$$
M_d = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}. \quad (11)
$$

The problem that we address in this paper is the derivation of conditions on $M$ and $V$ such that an explicit solution to (9)—satisfying (8)—can be obtained. For simplicity, we will take $q'' = 0$ and make the observation that, under Assumptions A1 and A3, the origin is an assignable equilibrium for (1) only if

$$
\nabla_1 V(0) = 0, \quad (12)
$$

that will be assumed in the sequel.

III. PRELIMINARY LEMMA

The following lemma, of interest on its own, lies at the core of our subsequent developments. It identifies a class of PDEs of the form (9), whose solution can be transformed into the solution of ordinary differential equations (ODEs).

Lemma 3.1: Consider the PDE in the unknown function $V_d : \mathbb{R}^2 \to \mathbb{R}$,

$$
a(q_2) \nabla_1 V_d + \nabla_2 V_d = b(q) \quad (13)
$$

where $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R}^2 \to \mathbb{R}$. Assume $b$ can be factored as

$$
b(q) = c(q_2) + B^\top (q_2) N(q_1) \quad (14)
$$

for some $c : \mathbb{R} \to \mathbb{R}$, $B : \mathbb{R} \to \mathbb{R}^p$ and $N : \mathbb{R} \to \mathbb{R}^p$ solution of the ODEs

$$
N' = AN, \quad (15)
$$

for some constant matrix $A \in \mathbb{R}^{p \times p}$ and initial conditions $N(0) \in \mathbb{R}^p$.

Then, for all differentiable functions $\Phi : \mathbb{R} \to \mathbb{R}$, a solution of (13) is given by

$$
V_d(q) = F(q_2) + H^\top (q_2) N(q_1) + \Phi(q_1 - z(q_2)), \quad (16)
$$

with $F : \mathbb{R} \to \mathbb{R}$, $H : \mathbb{R} \to \mathbb{R}^p$ and $z : \mathbb{R} \to \mathbb{R}$ solutions of the ODEs

$$
F' = c, \quad H' = -a A^\top H + B, \quad z' = a. \quad (17)
$$

Proof. The proof is obtained by direct substitution of (16) into (13). Indeed, differentiating (16)

$$
\nabla V_d = \begin{bmatrix} H^\top N' + \Phi' \\ F' + N^\top H' - a \Phi' \end{bmatrix} = \begin{bmatrix} H^\top AN + \Phi' \\ c + B^\top N - a \nabla_1 V_d \end{bmatrix}
$$

that, taking into account (14), clearly verifies (13). □□□

From (16), (17) it is clear that an explicit solution to the PDE (13) is available provided we can compute the integrals

$$
\int c(\xi) d\xi, \quad \int a(\xi) d\xi, \quad \int \Psi(\xi, q_2) B(\xi) d\xi,
$$

where $\Psi$ is the “state transition matrix” of the $H$ equation in (17)

$$
\Psi(q_2, \xi) = e^{-\int_{\xi}^{q_2} a(s) ds} A^\top.
$$

The evaluation of $\Psi$ can sometimes be simplified introducing a change of coordinates as follows. Assume the function $z$ is invertible, that is, there exists $q_2 : \mathbb{R} \to \mathbb{R}$ such that $z(q_2(s)) = s$ for all $s \in \mathbb{R}$. For all functions $x : \mathbb{R} \to \mathbb{R}$ introduce the following notation for the function compositions

$$
x(z) \triangleq x(q_2(z)).
$$

A simple application of the chain rule, i.e. $dH/dq_2 = \hat{a} \partial x/\partial q_2$, transforms (17) into

$$
\hat{F'} = \hat{c}, \quad \hat{H'} = -A^\top \hat{H} + \frac{\hat{B}}{\hat{a}}, \quad (18)
$$

where we notice that the function $a$ has been removed from the $\hat{H}$ equation, yielding a linear forced ODE, whose state transition matrix can be trivially calculated.

From the function $\hat{H}$ one can obtain the desired $H$ noting that

$$
\hat{H}(z(s)) = H((q_2(z(s)) = H(s), \forall s \in \mathbb{R}.
$$

2This assumption implies that the dependence of $b$ on $q_1$ consists of sine, cosine, exponential and polynomial functions generated as solutions of the linear ODEs (15).
IV. SOLVING THE POTENTIAL ENERGY PDE

In this section we use Lemma 3.1 and the change of coordinates procedure described above to give conditions on \( M \) and \( V \) such that an explicit solution to (9)—satisfying (8)—can be obtained.

We make the following assumptions:

A4. \( \nabla V(q) \) can be factored as

\[
\nabla V(q) = c_0(q) + B_0(q_2)N(q_1)
\]

(19)

for some \( c_0 : \mathbb{R} \to \mathbb{R}, B_0 : \mathbb{R} \to \mathbb{R}^p \) and \( N : \mathbb{R} \to \mathbb{R}^p \) solution of the ODE

\[
N' = AN
\]

(20)

for some constant matrix \( A \in \mathbb{R}^{p \times p} \) and \( N(0) \in \mathbb{R}^p \).

A5. \( k_1 > 0, k_2 \in \mathbb{R} \), elements of \( M_d \), are such that

\[
e_i^T M^{-1}(0) \left[ \begin{array}{c} k_1 \\ k_2 \end{array} \right] \neq 0, \quad i = 1, 2
\]

with \( e_i \in \mathbb{R}^2 \) the orthonormal Euclidean basis.

A6. The integral

\[
z(q_2) \triangleq \int_0^{q_2} \frac{\gamma_1(\sigma)}{\gamma_2(\sigma)} \, d\sigma,
\]

(21)

where \( \gamma \) is defined in (10), is computable and \( z(q_2) \) is a diffeomorphism with an inverse map \( \tilde{q}_2 : \mathbb{R} \to \mathbb{R} \), i.e., \( z(\tilde{q}_2(s)) = s, \forall s \in \mathbb{R} \). Furthermore, we can compute the functions

\[
\tilde{F}(z) \triangleq \int_0^z \frac{c_0}{\gamma_1}(\tilde{q}_2(\sigma)) \, d\sigma
\]

\[
\tilde{H}_0(z) \triangleq \int_0^z e^{(A^T \sigma)} B_0 \frac{\gamma_1}{\gamma_2}(\tilde{q}_2(\sigma)) \, d\sigma.
\]

(22)

**Proposition 4.1:** Assume \( M(q_2) \) and \( V(q) \) satisfy conditions A4–A6. Then the function

\[
V_d(q) = \tilde{F}(z(q_2)) + \nabla \nabla^T q_1 \tilde{H}(z(q_2)) + \Phi(q_1 - z(q_2))
\]

solves the potential energy PDE (9) for all differentiable functions \( \Phi : \mathbb{R} \to \mathbb{R} \) and all vectors \( \tilde{H}(0) \in \mathbb{R}^2 \).

**Proof:** The proof is a direct application of Lemma 3.1 and the change of coordinates described in the previous section. Indeed, under Assumption A5, \( \nabla V(q) \) can be written in the forms (13) and (14), respectively, with

\[
a \triangleq \frac{\gamma_1}{\gamma_2}, \quad b \triangleq \frac{\nabla q_1 V}{\gamma_2}, \quad c \triangleq \frac{c_0}{\gamma_2}, \quad B \triangleq \frac{B_0}{\gamma_2}.
\]

The ODEs (17), in the new coordinate \( z \), take the form (18) with

\[
\frac{\tilde{c}_2}{a}(z) = \frac{c_0}{\gamma_1}(\tilde{q}_2(z)), \quad \frac{\tilde{B}}{a}(z) = \frac{B_0}{\gamma_1}(\tilde{q}_2(z)).
\]

(24)

The solution of these ODEs, given in the proposition, completes the proof. \( \square \)

V. MAIN STABILIZATION RESULT

In the previous section we proposed a parametrization of the assignable potential energy functions in terms of the first column of \( M_d \) (the reals \( k_1 > 0, k_2 \in \mathbb{R} \)), the vector \( \tilde{H}(0) \in \mathbb{R}^2 \) and the (differentiable) function \( \Phi \). Here we will impose some additional constraints on these parameters to ensure stability of the closed-loop that, besides positivity of \( M_d \) (that is satisfied with a suitable choice of \( k_3 \)), requires assignment of the desired minimum to \( V_d \), i.e., (8). Towards this end we make the following assumptions.

A7. There exists \( k_1 > 0, k_2 \in \mathbb{R} \) such that

\[
\rho \triangleq [k_1 \quad k_2] M^{-1}(0) \begin{bmatrix} \nabla_{11} V(0) \\ \nabla_{12} V(0) \end{bmatrix} > 0.
\]

A8. \( \tilde{H}(0) \) and \( \Phi \) are such that

\[
\tilde{H}(0)AN(0) + \Phi'(0) = 0
\]

(25)

\[
\nabla(\nabla^T \tilde{H}(0)AN(0) + \Phi''(0)) > \frac{[\nabla_{11} V(0)]^2}{\rho}
\]

(26)

Assumption A8 is stated in this form for the sake of generality, but there always exist a vector \( \tilde{H}(0) \) such that it is satisfied with a quadratic function \( \Phi(s) = k_3 s^2 \), \( k_3 > 0 \). Indeed, in this case (25) holds taking \( \tilde{H}(0) \) orthogonal to \( AN(0) \). Furthermore, (26) will be satisfied taking \( k_3 \) sufficiently large. Assumption A7, on the other hand, imposes a real constraint on the admissible \( M \) and \( V \).

We are in position to present our main result, which essentially boils down to proving that, if A7 and A8 hold, \( V_d \) has an isolated local minimum at zero.

**Proposition 5.1:** Consider the system (1), (2) satisfying Assumptions A1–A6. Then the IDA–PBC law (5) takes the form

\[
u = \frac{1}{2} \nabla_2 (p^T M^{-1} p) + \nabla_2 V - [k_2 \quad k_3] M^{-1} \nabla V_d
\]

and it yields the closed-loop dynamics

\[
\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1} M_d \\ -M_d M^{-1} & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix},
\]

where \( H_d(q, p) \) is given by (7) with \( M_d \) the constant matrix (11) and \( V_d(q) \) as in Proposition (4.1).

Furthermore, if \( k_3 > \frac{k_2^2}{4} \) and Assumptions A7 and A8 hold, the origin is a stable equilibrium of (27) with Lyapunov function \( H_d \).

**Proof:** The control expression follows from (5) taking into account Assumptions A1–A3, and the fact that \( J_2 = 0 \) and \( M_d \) is constant.

Proposition 4.1 establishes that, under Assumptions A4–A6, the proposed \( V_d \) solves the potential energy PDE (8). Now, the condition on \( k_3 \) ensures \( M_d > 0 \), hence \( H_d \) will be a Lyapunov function for the system if \( V_d \) has an isolated local minimum at zero. To prove this fact we will find
convenient to study the function in the coordinates \((q_1, z)\), hence we define
\[
\hat{V}_d(q_1, z) \triangleq \hat{F}(z) + N^\top(q_1)\hat{H}(z) + \Phi(q_1 - z)
\]
\[
\tilde{V}(q_1, z) \triangleq V(q_1, \hat{q}_2(z)).
\]

The gradient of \(\tilde{V}_d\) is
\[
\nabla \tilde{V}_d = \left[ \begin{array}{c} \tilde{H}^\top A N + \Phi' \\ \tilde{F}' + N^\top \tilde{H}' - \Phi' \end{array} \right] = \left[ \begin{array}{c} \tilde{H}^\top A N + \Phi' \\ \tilde{F}' + N^\top \tilde{H}' - \Phi' \end{array} \right],
\]
where the second identity is obtained using (18). Now, from (19) and (24) we have that
\[
\frac{\tilde{c}}{\tilde{a}}(z) + N^\top(q_1)\tilde{B} = \frac{1}{\gamma_1(z)} \nabla \tilde{V}(q_1, z)
\]
which, in view of (12) and the fact that \(\hat{q}_2(0) = 0\), is zero at \((q_1, z) = (0, 0)\). Consequently, \(\nabla \tilde{V}_d(0) = 0\) if and only if (25) holds.

Some simple calculation show that the Hessian of \(\tilde{V}_d\) is given by
\[
\nabla^2 \tilde{V}_d(q_1, z) = \begin{bmatrix} \kappa_1 & \kappa_2 - \kappa_1 \\ \kappa_2 - \kappa_1 & \kappa_1 - \kappa_2 + \kappa_3 \end{bmatrix},
\]
where the real-valued functions \(\kappa_i, i = 1, \ldots, 3\) are defined as
\[
\kappa_1 = \tilde{H}^\top A^2 N + \Phi'' \\
\kappa_2 = N^\top A^\top \tilde{B} = \frac{1}{\gamma_1} \nabla_{11} \tilde{V} \\
\kappa_3 = \nabla_2 \frac{1}{\gamma_1} \nabla_{11} \tilde{V} = \frac{1}{\gamma_1^2} \left( \gamma_2 \nabla_{12} \tilde{V} - \frac{\gamma_2}{\gamma_1} \nabla_{11} \tilde{V} \right).
\]
The Hessian is positive-definite if \((\kappa_2 + \kappa_3) > 0\) and \(\kappa_1 > -\frac{\kappa_2^2}{\kappa_3} \geq 0\). Evaluated at the origin, and taking into account that \(\nabla_{11} \tilde{V}(0) = 0\), we have
\[
\frac{\kappa_1(0)}{\kappa_2(0) + \kappa_3(0)} = \frac{\nabla_{11} \tilde{N}(0)}{\rho}
\]
with \(\rho\) as defined in Assumption A7, completing the proof.

VI. THE ACROBOT EXAMPLE

In this section we prove that the methodology described above applies to the interesting Acrobot system described in [10]. The equations of motion of the Acrobot (schematic shown in Figure 1) are given by (1), (2) with \(n = 2, m = 1\),
\[
M(q_2) = \begin{bmatrix} c_1 + c_2 + 2c_3 \cos q_2 & c_2 + c_3 \cos q_2 \\ c_2 + c_3 \cos q_2 & c_2 \end{bmatrix},
\]
\[
V(q) = g[c_4 \cos q_1 + c_5 \cos(q_1 + q_2)],
\]
\[
G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T,
\]
with \(g\) the gravitational constant and the parameters
\[
c_1 = m_1 l_1^2 + m_2 l_2^2 + I_1, c_2 = m_2 l_1^2 + I_2
\]
\[
c_3 = m_2 l_2 c_2 \\
c_4 = m_1 l_2 c_1 + m_2 l_1, c_5 = m_2 l_2
\]
verifying \(c_1 c_2 > c_3^2\).

The control objective is to stabilize the upward equilibrium position, i.e., \(q^* = (0, 0)\).

A. Verifying the Assumptions

The Acrobot clearly satisfies Assumptions A1–A3. Assumption A4 also holds with
\[
c_0(q_2) = 0, B_0(q_2) = - \begin{bmatrix} c_4 g + c_5 g \cos q_2 \\ c_3 g \sin q_2 \end{bmatrix},
\]
\[
N(q_1) = \begin{bmatrix} \sin q_1 \\ \cos q_1 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, N(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

To verify Assumptions A5 and A6 we compute the vector \(\gamma\) given in (10) that, using (11), yields
\[
\gamma(q_2) = \frac{1}{\delta(q_2)} \begin{bmatrix} (k_1 - k_2) c_2 - k_2 c_3 \cos q_2 \\ k_2 (c_1 + c_2) - k_1 c_2 + c_3 (2k_2 - k_1) \cos q_2 \end{bmatrix}
\]
where \(\delta(q_2) \triangleq c_1 c_2 - c_3^2 \cos^2(q_2) > 0\). The ratio \(\frac{\gamma_2}{\gamma_1}\) can be expressed as
\[
\frac{\gamma_1}{\gamma_2} = \frac{a_3 + a_4 \cos q_2}{a_1 + a_2 \cos q_2},
\]
where
\[
a_1 \triangleq k_2 (c_1 + c_2) - k_1 c_2
\]
\[
a_2 \triangleq (2k_2 - k_1) c_3
\]
\[
a_3 \triangleq (k_1 - k_2) c_2
\]
\[
a_4 \triangleq -k_2 c_3.
\]

Even though the function \(\frac{\gamma_1}{\gamma_2}\) above can be explicitly integrated, to simplify the derivations we exploit the fact that the ratio takes a constant value \(\mu \triangleq \frac{a_4}{a_2}\) when \(a_3 a_2 =\)
The latter fixes the following relationship between the free parameters \(k_1\) and \(k_2\)
\[
k_1 = (1 \pm \sqrt{\frac{c_1}{c_2}})k_2, \tag{28}
\]
where we can select either the positive or the negative sign, and the ratio takes the value
\[
\frac{\gamma_1}{\gamma_2} = \mu \triangleq \pm \frac{1}{\sqrt{c_1/c_2} - 1}.
\]
From (28) we see that, since \(k_1 > 0\), our only choice when \(c_1 = c_2\) is the positive sign but, unfortunately, \(\mu\) is not defined in this case, hence we need the following:

**Assumption A6'** \(c_1 \neq c_2\).

Under Assumption A6' (21) becomes \(z(q_2) = \mu q_2\), which is obviously a diffeomorphism. Interestingly, for the choice of parameters (28), Assumption A5 is satisfied for all \(q_2\). Indeed, replacing (28) in (27) and doing some simple calculations we get
\[
\gamma(q_2) = \frac{k_2(\pm \sqrt{c_1 c_2} - c_3 \cos q_2)}{\delta(q_2)} \left[ \pm \frac{1}{\sqrt{c_1/c_2} - 1} \right]
\]
which, in view of Assumption A6', \(c_1 c_2 > c_3^2\) and \(\delta > 0\), is nonzero for all \(q_2\).

After some lengthy, but straightforward, calculations we see that the remaining integrability conditions of Assumption A6 are easily verifiable with \(\bar{F} = 0\) and \(\bar{H}\) as in (23) with
\[
e^{-A^\top z} = \begin{bmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{bmatrix}
\]
\[
\bar{H}_0^0(z) = \begin{bmatrix} H_1^0(z) \\ H_2^0(z) \end{bmatrix}
\]
where
\[
\bar{H}_0^0(z) = -b_1 \sin z - b_3 \sin(\pm \sqrt{\frac{c_1}{c_2}} z)
\]
\[-b_3 \sin((\pm 2 \sqrt{\frac{c_1}{c_2}} - 1) z) + b_4 \sin((\pm \sqrt{\frac{c_1}{c_2}} - 2) z)
\]
\[
\bar{H}_2^0(z) = b_1 \cos z + b_2 \cos(\pm \sqrt{\frac{c_1}{c_2}} z) + b_3 \cos((\pm 2 \sqrt{\frac{c_1}{c_2}} - 1) z) + b_4 \cos((\pm \sqrt{\frac{c_1}{c_2}} - 2) z)
\]
and the constants \(b_i, i = 1 \ldots 4\) are defined as
\[
b_1 = \frac{1}{k_2} g \left( \pm \sqrt{c_1 c_2} + \frac{c_3 c_5}{2} \right)
\]
\[
b_2 = \frac{1}{k_2} g \mu \left( \frac{1}{2} c_3 c_4 + \sqrt{c_1 c_2} c_5 \right)
\]
\[
b_3 = \frac{1}{k_2} g c_3 c_5 \mu
\]
\[
b_4 = \frac{1}{2} g \frac{c_3 c_4}{\mu} - \frac{1}{\mu - 1},
\]
where we have pulled-out the factor \(\frac{1}{k_2}\) to underscore its role as a “scaling gain” for \(V_d\).

In the coordinates \((q_1, q_2)\), the potential energy \(V_d\) takes the simple form
\[
V_d(q_1, q_2) = N^\top(q_1) e^{A(q_2)} \bar{H}(0) + b_1 \cos q_1 + b_2 \cos(q_1 + q_2) + b_3 \cos(q_1 + 2q_2) + b_4 \cos(q_1 - q_2) + \Phi(q_1 - \mu q_2).
\]

Once we have established, following Proposition 4.1, the existence of a solution for the potential energy PDE, it remains to verify the stability conditions of Proposition 5.1. For, using (28), we compute
\[
\rho = -\frac{g k_2}{c_1 c_2 - c_3^2} \left( -c_3 \pm \sqrt{c_1 c_2} c_4 - c_5 \left( -c_1 \pm \sqrt{c_1/c_2} \right) \right),
\]
hence, Assumption A7 is equivalent to

**Assumption A7'** The constants \(k_2\) and \(c_i\) are such that
\[
k_2 \left( -c_3 \pm \sqrt{c_1 c_2} c_4 - c_5 \left( -c_1 \pm \sqrt{c_1/c_2} \right) \right) < 0
\]
Assumption A6' is generically satisfied and the only “real” assumption is A7'. However, note that—for the case when \(c_1 = c_2\)—the sign of \(k_2\) is not fixed, hence it is always possible to fulfill this condition.

Finally, Assumption A8 can be shown to hold if and only if

**Assumption A8'** The elements of the vector \(\bar{H}(0)\) and the function \(\Phi\) satisfy
\[
\bar{H}_1(0) = -\Phi'(0), \quad \bar{H}_2(0) < \Phi''(0) - \frac{g^2 (c_4 + c_5)^2}{\rho}.
\]

### B. Asymptotic Stability

We are in position to state our stabilization result for the Acrobot where, to achieve asymptotic stability, we have added a damping injection term [9]. To simplify the statement we will take \(k_2 > 0\) and select the positive sign in (28).

**Proposition 6.1:** Consider the Acrobot system satisfying Assumptions A6', A7' (with the positive sign and \(k_2 > 0\)) and A8'. Fix
\[
k_1 = (1 + \frac{c_3}{c_2})k_2, \quad k_3 > \frac{k_2}{1 + \frac{c_3}{c_2}}.
\]
Let \( \Phi(q_1 - \mu q_2) = \frac{k_p}{2} (q_1 - \mu q_2)^2 \) with
\[
\mu = \frac{1}{\sqrt{\frac{c_2}{c_1} - 1}}, \quad k_p > \frac{q^2(c_4 + c_5)^2}{\rho} + \hat{H}_2(0),
\]
with \( \rho \) given by (30) (with positive sign), \( \hat{H}_2(0) \) arbitrary and \( \hat{H}_1(0) = 0 \). Then the (globally defined) IDA–PBC law
\[
u = \frac{1}{2} \nabla_2 \left( (p^T M^{-1} p) - \left[ k_2 \quad k_3 \right] M^{-1} \nabla V_d + \nabla_2 V \right) + \frac{k_c}{k_1 k_3 - k_2^2} (k_2 p_1 - k_1 p_2),
\]
where \( k_c > 0 \) is the damping injection gain and
achieve (local) asymptotic stabilization of the trivial equilibrium with Lyapunov function \( H_d \). An estimate of the domain of attraction is given by \( \Omega_c \) where \( \Omega_c = \{(q, p) \in \mathbb{R}^4 \mid H_d(q, p) < c\} \)

\[
e^c = \sup\{c > H_d(q_*, 0) \mid \Omega_c \text{ is bounded}\}
\]

**Proof:** It is easy to verify that, with the given selection of \( \Phi \) and \( \hat{H}(0) \), Assumption A8’ is satisfied. Henceforth, stability follows from Proposition 5.1 and it only remains to prove the asymptotic part.

The derivative of \( H_d \) along the trajectories of the closed-loop system is given by
\[
\dot{H}_d = -k_1 \gamma_1 (\gamma_1 M_d^{-1} p)^2 \leq 0,
\]
with \( \gamma_1 \) the Euclidean norm. Define the residual set as
\[
\Pi = \{(q, p) \in \Omega_c \mid \gamma_1 M_d^{-1} p = 0\}.
\]
Let \((q, p) \in \Pi\). We have
\[
0 = G^T M_d^{-1} p = G^T M_d^{-1} M q \Rightarrow \dot{q}_1 = \frac{\gamma_1}{\gamma_2} \dot{q}_2 = \mu q_2.
\]
Integrating (31), we obtain \( q_1(t) = \mu q_2(t) + C \), where \( C \) is the integration constant. Since the origin belongs to the residual set, the constant \( C \) equals zero. Hence the trajectories in \( \Pi \) are restricted to the line \( q_1(t) = \mu q_2(t) \).

Finally, differentiating the left-hand side of (31), we have
\[
0 = G^T M_d^{-1} \dot{p} = -G^T M_d^{-1} \nabla V_d
\]
where, to obtain the second identity, we have used the fact that the closed-loop dynamics verifies
\[
\dot{\nu} = -M_d M_d^{-1} \nabla_q V_d(q) - k_c G G^T M_d^{-1} p.
\]

Now
\[
-G^T M_d^{-1} \nabla V_d(q_2) = 0
\]
\[
[ -c_2 + c_3 \cos q_2 \quad c_1 + c_2 + 2c_3 \cos q_2 \] \nabla V_d(q_2) = 0.
\]
\[
\therefore
\]

The row vector on the left has entries with even functions, while \( \nabla V_d \) is composed of odd functions. The inner-product above is then a product of even and odd functions, which is an odd function with an isolated zero at \( q_2 = 0 \). Hence the only trajectory that can identically stay in \( \Pi \) is \( (q(t), p(t)) = (0, 0) \) and hence by La Salle’s theorem, the origin is locally asymptotically stable. Finally, the estimate of the domain of attraction follows from the fact that \( \Omega_c \) is the largest bounded sub-level set of \( H_d \).

\[\square\square\square\]

**C. Simulations**

The Acrobot parameters—resulting from \( m_1 = 4, m_2 = 1, I_1 = 0.3333, I_2 = 1.3333, l_1 = 1, l_2 = 4, e_{c1} = 0.5000, e_{c2} = 2 \)—are displayed in Table I

<table>
<thead>
<tr>
<th>c1</th>
<th>c2</th>
<th>c3</th>
<th>c4</th>
<th>c5</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.3333</td>
<td>5.3333</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

**TABLE I**

**ACROBOT PARAMETERS**

In (28) we select the positive sign, \( k_2 = 1 \) and \( k_3 = 5.9073 \) yielding
\[
M_d = \begin{bmatrix} 0.3386 & 1 \\ 1 & 5.9073 \end{bmatrix} > 0, \quad \mu = -0.6019.
\]

The level sets of \( V_d \), for \( k_p = 279.8004, \hat{H}_2(0) = 10 \), are shown in Fig. 2. We observe from the figure that there are closed sets containing points in the lower half plane of the configuration space, that is, \( |q_1| > \pi/2 \). This ensures that, starting with zero velocities, the IDA–PBC will swing–up the Acrobot from the lower half plane and at the same time asymptotically stabilize its upward equilibrium. To the best of our knowledge this is the first continuous controller that achieves these two objectives for the Acrobot. It should, however, be pointed out that our ability to enlarge the domain of attraction is severely stymied as the actual shape of \( V_d \) is essentially determined by the robot parameters. Indeed, from (29) and (29) it is clear that the effect of the design parameters at our disposal (\( \hat{H}_2(0), k_2, k_3 \)) “disappears” along the line \( q_1 = \mu q_2 \) where—besides the locally asymptotically stable zero equilibrium—the function \( V_d \) might exhibit other extrema that generate spurious equilibria for the closed-loop system.

The time-response of the system starting from the fully stretched downward position, i.e., \( (q(0), p(0)) = (-\pi + 0.001, 0, 0, 0) \) with \( k_c = 12 \) is depicted in Fig.

\[assuming a quadratic \Phi\]
3. Although this initial condition does not belong to the estimated domain of attraction \( \Omega_0 \), the simulation shows that the controller effectively swings–up the acrobot from the downward position. In order to justify theoretically this simulated evidence, current research is under way to improve our estimate of the basin of attraction.

Fig. 2. Level sets of \( V_d \)

Fig. 3. Transient responses

VII. CONCLUSIONS

In the frame-work of IDA-PBC, we have presented a methodology to reduce the potential PDE into an ODE for underactuated mechanical systems with an underactuation degree one, thus overcoming the main stumbling block in its application. The solvability of the ODES’s is demonstrated on the acrobot, wherein the control task of swing-up and balance about the upward equilibrium point is achieved.

ACKNOWLEDGEMENTS

The work of Alessandro Astolfi, Giuseppe Viola and Arun Mahindrakar were partially sponsored by the Leverhulme Trust, the European Community Marie Curie Fellowship (in the framework of the CTS, contract number: HPMT-CT-2001-00278) and the European project Geoplex (reference code IST-2001-34166), respectively.

REFERENCES