EE611 Solutions to Problem Set 2

1. (a) The minimum distance between any two points in the signal constellation is

$$d_{min} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{3}.$$

(b) The decision regions for the optimal receiver are shown in Figure 1.

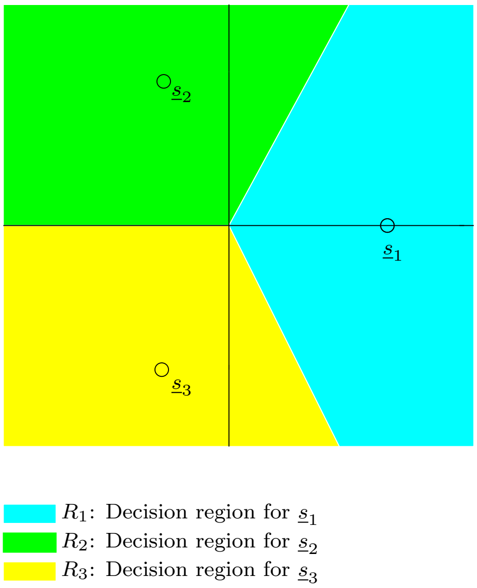

Figure 1: Decision Regions

(c) $P[\varepsilon]_{min.} = P[\varepsilon|\underline{s}_1]P[\underline{s}_1] + P[\varepsilon|\underline{s}_2]P[\underline{s}_2] + P[\varepsilon|\underline{s}_3]P[\underline{s}_3]$. Since the symbols are equally likely, we have $P[\varepsilon]_{min.} = \frac{1}{3}\left[P[\varepsilon|\underline{s}_1] + P[\varepsilon|\underline{s}_2] + P[\varepsilon|\underline{s}_3]\right]$. Because of the symmetry in the signal constellation, we have $P[\varepsilon]_{min.} = P[\varepsilon|\underline{s}_1]$.

(d) $P[\varepsilon]_{min.} = P[\varepsilon|\underline{s}_1] = P[\underline{r} \in R_2|\underline{s}_1] + P[\underline{r} \in R_3|\underline{s}_1]$. We know that

$$P[\underline{r} \in R_2|\underline{s}_1] < Q\left(\frac{\sqrt{3}}{\sqrt{2N_0}}\right),$$

and

$$P[\underline{r} \in R_3|\underline{s}_1] < Q\left(\frac{\sqrt{3}}{\sqrt{2N_0}}\right).$$

Therefore, we have

$$P[\varepsilon]_{min.} < 2Q\left(\frac{\sqrt{3}}{\sqrt{2N_0}}\right).$$

2. (i) As discussed in class, r_2 is relevant because r_2 is not independent of s given r_1.

(ii) For a given received vector $\underline{r} = \underline{\rho}$ (i.e., $[r_1 \quad r_2]^T = [\rho_1 \quad \rho_2]^T$), the optimal receiver chooses $\hat{s} = s_k$ such that $P[s = s_i | r_1 = \rho_1, r_2 = \rho_2]$ is maximized for $i = k$. Equivalently, $f_{r_1,r_2}(\rho_1, \rho_2 | s = s_i) P[s = s_i]$ is maximized for $i = k$. In this case, we have $P[s = s_0] = P[s = s_1] = 0.5$. Therefore, the optimal decision rule is $\hat{s} = \sqrt{E_s}$ if

$$f_{r_1,r_2}(\rho_1, \rho_2 | s = \sqrt{E_s}) > f_{r_1,r_2}(\rho_1, \rho_2 | s = -\sqrt{E_s}),$$

and $\hat{s} = -\sqrt{E_s}$ otherwise.

Given $s = \sqrt{E_s}$, $\underline{r}$ is a jointly Gaussian random vector with mean $\underline{m}_0 = [\sqrt{E_s} \;\; 0]^T$ and covariance matrix

$$C = \sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Given $s = -\sqrt{E_s}$, $\underline{r}$ is a jointly Gaussian random vector with mean $\underline{m}_1 = [-\sqrt{E_s} \;\; 0]^T = -\underline{m}_0$ and covariance matrix

$$C = \sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Now, the decision rule can be written as follows: $\hat{s} = \sqrt{E_s}$ if

$$\frac{1}{2\pi |C|^{1/2}} \exp\left\{-\frac{1}{2}(\underline{\rho} - \underline{m}_0)^T C^{-1} (\underline{\rho} - \underline{m}_0)\right\} > \frac{1}{2\pi |C|^{1/2}} \exp\left\{-\frac{1}{2}(\underline{\rho} + \underline{m}_0)^T C^{-1} (\underline{\rho} + \underline{m}_0)\right\}$$

and $\hat{s} = -\sqrt{E_s}$ otherwise. Equivalently, we have $\hat{s} = \sqrt{E_s}$ if

$$(\underline{\rho} - \underline{m}_0)^T C^{-1} (\underline{\rho} - \underline{m}_0) < (\underline{\rho} + \underline{m}_0)^T C^{-1} (\underline{\rho} + \underline{m}_0) \tag{1}$$

and $\hat{s} = -\sqrt{E_s}$ otherwise. Now, (1) can be simplified as follows:

$$\underline{\rho}^T C^{-1} \underline{\rho} - \underline{\rho}^T C^{-1} \underline{m}_0 - \underline{m}_0^T C^{-1} \underline{\rho} + \underline{m}_0^T C^{-1} \underline{m}_0 < \underline{\rho}^T C^{-1} \underline{\rho} + \underline{\rho}^T C^{-1} \underline{m}_0 + \underline{m}_0^T C^{-1} \underline{\rho} + \underline{m}_0^T C^{-1} \underline{m}_0$$

$$-\underline{\rho}^T C^{-1} \underline{m}_0 - \underline{m}_0^T C^{-1} \underline{\rho} < \underline{\rho}^T C^{-1} \underline{m}_0 + \underline{m}_0^T C^{-1} \underline{\rho}$$

This inequality simplifies to

$$\underline{\rho}^T C^{-1} \underline{m}_0 > 0,$$

or

$$2\rho_1 - \rho_2 > 0.$$

Therefore, the optimal decision rule is: $\hat{s} = \sqrt{E_s}$ if

$$2\rho_1 - \rho_2 > 0.$$

and $\hat{s} = -\sqrt{E_s}$ otherwise.

(iii) For a given received vector $\underline{r} = \underline{\rho}$ (i.e., $[r_1 \quad r_2]^T = [\rho_1 \quad \rho_2]^T$), the optimal receiver chooses $\hat{s} = s_k$ such that $P[s = s_i | r_1 = \rho_1, r_2 = \rho_2]$ is maximized for $i = k$. Equivalently, $f_{r_1,r_2}(\rho_1, \rho_2 | s = s_i) P[s = s_i]$ is maximized for $i = k$. In this case, we have $P[s = s_0] = P[s = s_1] = 0.5$. Therefore, the optimal decision rule is $\hat{s} = \sqrt{E_s}$ if

$$f_{r_1,r_2}(\rho_1, \rho_2 | s = \sqrt{E_s}) > f_{r_1,r_2}(\rho_1, \rho_2 | s = -\sqrt{E_s}),$$

and $\hat{s} = -\sqrt{E_s}$ otherwise.

Given $s = \sqrt{E_s}$, $\underline{r}$ is a jointly Gaussian random vector with mean $\underline{m}_0 = [\sqrt{E_s} \;\; 2\sqrt{E_s}]^T$ and covariance matrix

$$C = \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Given $s = -\sqrt{E_s}$, $\underline{r}$ is a jointly Gaussian random vector with mean $\underline{m}_1 = [-\sqrt{E_s} \;\; -2\sqrt{E_s}]^T = -\underline{m}_0$ and covariance matrix

$$C = \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

As in (ii), the decision rule can be written as follows: $\hat{s} = \sqrt{E_s}$ if

$$\underline{\rho}^T C^{-1} \underline{m}_0 > 0,$$

or

$$\underline{\rho}^T \underline{m}_0 > 0,$$

$$\rho_1 + 2\rho_2 > 0.$$

Therefore, the optimal decision rule is: $\hat{s} = \sqrt{E_s}$ if

$$\rho_1 + 2\rho_2 > 0.$$

and $\hat{s} = -\sqrt{E_s}$ otherwise.

3. If one of two equally likely messages $s_0 = -2$ and $s_1 = 2$ is transmitted, the optimum receiver yields $P[\varepsilon] = 0.01$, i.e.,

$$Q\left(\frac{2}{\sigma}\right) = 0.01,$$

where σ is the standard deviation of AWGN.

(i) (a) When one of three equally likely messages $s_0 = -4$, $s_1 = 0$, and $s_2 = 4$, is transmitted, then

$$\begin{aligned} P[\varepsilon]_{min.} &= \tfrac{1}{3}\left[P[\varepsilon|s_0] + P[\varepsilon|s_1] + P[\varepsilon|s_2]\right] \\ &= \tfrac{1}{3}\left[Q\left(\tfrac{2}{\sigma}\right) + 2Q\left(\tfrac{2}{\sigma}\right) + Q\left(\tfrac{2}{\sigma}\right)\right] \\ &= \tfrac{4}{3}Q\left(\tfrac{2}{\sigma}\right) \\ &= 0.0133. \end{aligned}$$

(b) When one of four equally likely messages $s_0 = -4$, $s_1 = 0$, $s_2 = 4$, and $s_3 = 8$, is transmitted, then

$$\begin{aligned} P[\varepsilon]_{min.} &= \tfrac{1}{4}\left[P[\varepsilon|s_0] + P[\varepsilon|s_1] + P[\varepsilon|s_2] + P[\varepsilon|s_3]\right] \\ &= \tfrac{1}{4}\left[Q\left(\tfrac{2}{\sigma}\right) + 2Q\left(\tfrac{2}{\sigma}\right) + 2Q\left(\tfrac{2}{\sigma}\right) + Q\left(\tfrac{2}{\sigma}\right)\right] \\ &= \tfrac{3}{2}Q\left(\tfrac{2}{\sigma}\right) \\ &= 0.015. \end{aligned}$$

(ii) The error probabilities calculated in (i) do not change. The decision regions change (shift by 1).

4. We need to determine the probability of symbol error of QPSK given that the symbols are equally likely. Because of the symmetry of the signal constellation, we have $P[\varepsilon] = P[\varepsilon|\mathbf{s}_0] = 1 - P[\text{correct decision}|\mathbf{s}_0]$.

$$P_e = 1 - \left\{1 - Q\left(\frac{d}{2\sigma}\right)\right\}^2$$

where $d = \sqrt{2}$, and $\sigma = \sqrt{\dfrac{N_0}{2}}$, i.e.$\dfrac{d}{2\sigma} = \sqrt{\dfrac{1}{N_0}}$.

5. Using $s_1(t)$, $s_2(t)$ and $s_4(t)$ as the three orthonormal basis functions, we can represent the signals in vector form as follows: $\underline{s}_0 = (0\ \ 0\ \ 0)$, $\underline{s}_1 = (1\ \ 0\ \ 0)$, $\underline{s}_2 = (0\ \ 1\ \ 0)$, $\underline{s}_3 = (1\ \ 1\ \ 0)$, $\underline{s}_4 = (0\ \ 0\ \ 1)$, $\underline{s}_5 = (1\ \ 0\ \ 1)$, $\underline{s}_6 = (0\ \ 1\ \ 1)$, $\underline{s}_7 = (1\ \ 1\ \ 1)$. These are the vertices of a cube in three-dimensional space. The probability of error given any transmitted message is the same irrespective of the message (symmetry in the constellation) and is equal to the overall probability of error (equally likely messages). Thus the probability of error is determined as follows (as 1 - probability of correct decision):

$$P_e = 1 - \left\{1 - Q\left(\frac{d}{2\sigma}\right)\right\}^3,$$

where $d = 1$ and $\sigma = \sqrt{N_0/2}$.