Solutions to Problem Set 7

EE419: Digital Communication Systems

Check the solutions for possible bugs!

1. The received signal $z[k] = cs[k] + n[k]$, where $n[k] \sim N(0, N_0)$ is white Gaussian and $s[k] \in \{-1, 1\}$ is *iid* uniform independent of $n[k]$. The i-th entry of $\underline{\alpha}$ is given by $\alpha_i = \mathrm{E}[s[k]z^*[k+i]]$ (the index i runs from P to $-P$).

$$\alpha_i = \mathrm{E}[c^* s[k]s^*[k+i] + s[k]n^*[k+i]] = \begin{cases} c^* & i = 0 \\ 0 & i \neq 0 \end{cases}$$

The (i,l)-th entry of ϕ is given by $\phi_{il} = \mathrm{E}[z^*[k+i]z[k+l]]$ (the indices i and l run from P to $-P$).

$$\phi_{il} = \mathrm{E}[(c^* s^*[k+i] + n^*[k+i])(cs[k+l] + n[k+l])] = \begin{cases} |c|^2 + N_0 & i = l \\ 0 & i \neq l \end{cases}$$

Hence,

$$\underline{c}_{\text{opt}} = \begin{bmatrix} 0 \\ \vdots \\ c^*/(|c|^2 + N_0) \\ \vdots \\ 0 \end{bmatrix}$$

and MSE$=1 - \dfrac{|c|^2}{|c|^2 + N_0} = \dfrac{N_0}{|c|^2 + N_0}$ for all P.

2. The received signal $z[k] = s[k] + cs[k-1] + n[k]$, where $n[k] \sim N(0, N_0)$ is white Gaussian and $s[k] \in \{-1, 1\}$ is *iid* uniform independent of $n[k]$. The i-th entry of $\underline{\alpha}$ is given by $\alpha_i = \mathrm{E}[s[k]z^*[k+i]]$ (the index i runs from P to $-P$).

$$\alpha_i = \mathrm{E}[s[k]s^*[k+i] + c^* s[k]s^*[k+i-1] + s[k]n^*[k+i]] = \begin{cases} 1 & i = 0 \\ c^* & i = 1 \\ 0 & \text{else} \end{cases}$$

The (i,l)-th entry of ϕ is given by $\phi_{il} = \mathrm{E}[z^*[k+i]z[k+l]]$ (the indices i and l run from P to $-P$).

$$\phi_{il} = \mathrm{E}[(s^*[k+i] + c^* s^*[k+i-1] + n^*[k+i])(s[k+l] + cs[k+l-1] + n[k+l])] = \begin{cases} 1 + |c|^2 + N_0 & i = l \\ c^* & i = l+1 \\ c & i = l-1 \\ 0 & \text{else} \end{cases}$$

For $P = 0$, $\underline{c}_{\text{opt}} = 1/(1 + |c|^2 + N_0)$ and MSE$=1 - \dfrac{1}{1 + |c|^2 + N_0} = \dfrac{|c|^2 + N_0}{1 + |c|^2 + N_0}$. For $P = 1$,

$$\underline{c}_{\text{opt}} = \begin{bmatrix} 1 + |c|^2 + N_0 & c^* & 0 \\ c & 1 + |c|^2 + N_0 & c^* \\ 0 & c & 1 + |c|^2 + N_0 \end{bmatrix}^{-1} \begin{bmatrix} c^* \\ 1 \\ 0 \end{bmatrix}$$

with a similar formula for MSE. The case $P = 2$ is omitted.

3. The received signal $z[k] = \sum_{m=0}^{\infty}(-c)^m s[k-m] + n[k]$, where $n[k] \sim N(0, N_0)$ is white Gaussian and $s[k] \in \{-1, 1\}$ is *iid* uniform independent of $n[k]$. The i-th entry of $\underline{\alpha}$ is given by $\alpha_i = \mathrm{E}[s[k]z^*[k+i]]$ (the index i runs from P to $-P$).

$$\alpha_i = \mathrm{E}\left[\sum_{m=0}^{\infty}(-c^*)^m s[k]s^*[k+i-m] + s[k]n^*[k+i]\right] = \begin{cases}(-c^*)^i & i \geq 0\\ 0 & \text{else}\end{cases}$$

The (i,l)-th entry of ϕ is given by $\phi_{il} = \mathrm{E}[z^*[k+i]z[k+l]]$ (the indices i and l run from P to $-P$).

$$\begin{aligned}\phi_{il} &= \mathrm{E}\left[\left(\textstyle\sum_{m_1=0}^{\infty}(-c^*)^{m_1}s^*[k+i-m_1] + n^*[k+i]\right)\left(\textstyle\sum_{m_2=0}^{\infty}(-c)^{m_2}s[k+l-m_2] + n[k+l]\right)\right]\\ &= \begin{cases}\dfrac{1}{1-|c|^2} + N_0 & i = l\\ \dfrac{(-c^*)^d}{1-|c|^2} & i = l+d, d = 1, 2, \cdots\\ \dfrac{(-c)^d}{1-|c|^2} & i = l-d, d = 1, 2, \cdots\end{cases}\end{aligned}$$

For $P = 0$, $\underline{c}_{\text{opt}} = (1-|c|^2)/(1+N_0-N_0|c|^2)$ and MSE$=1-\dfrac{1-|c|^2}{1+N_0-N_0|c|^2}$. For $P = 1$ and $P = 2$, substitute the above values into the fomulae.

4. Suppose $H(z) = \sum_{l=-M}^{M} h_l z^{-l}$ resulting in $z[k+i] = \sum_{l=-M}^{M} h[l]s[k+i-l] + n[k+i]$. Let $b[k+i] = \sum_{l=-M}^{M} h[l]s[k+i-l]$ be the symbol component in $z[k+i]$. We see that

$$\begin{bmatrix} b[k+P]\\ \vdots\\ b[k]\\ \vdots\\ b[k-P]\end{bmatrix} = \begin{bmatrix} h[-M] & \cdots & \cdots & h[0] & \cdots & h[M] & 0 & \cdots & 0\\ 0 & h[-M] & \cdots & \cdots & h[0] & \cdots & h[M] & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & \cdots & 0 & h[-M] & \cdots & h[0] & \cdots & h[M] & 0\\ 0 & \cdots & \cdots & 0 & h[-M] & \cdots & h[0] & \cdots & h[M]\end{bmatrix}\begin{bmatrix} s[k+P+M]\\ \vdots\\ s[k+P]\\ \vdots\\ s[k]\\ \vdots\\ s[k-P]\\ \vdots\\ s[k-P-M]\end{bmatrix}.$$

Denote the $2P+1 \times 2P+2M+1$ matrix above as H. The symbol component after filtering $\{z[k]\}$ by $C(z) = \sum_{l=-P}^{P} c_l z^{-l}$ is given by

$$x[k] = \begin{bmatrix} c[-P] & \cdots & c[o] & \cdots & c[P]\end{bmatrix} H \begin{bmatrix} s[k+P+M]\\ \vdots\\ s[k+P]\\ \vdots\\ s[k]\\ \vdots\\ s[k-P]\\ \vdots\\ s[k-P-M]\end{bmatrix}. \tag{1}$$

In a ZF-LE, we choose $\underline{c} = [c[-P]\cdots c[0]\cdots c[P]]$ so that

$$x[k] = s[k] + \sum_{d<|l|\leq P+M} \alpha_l s[k+l] + \text{noise}$$

for a suitable d. Notice that the interference from $s[k+l]$, $1 \leq |l| \leq d$ is expected to be nulled by the equalizer. Let $\underline{h}_l$ denote the l-th column of H, whose columns are indexed from $-P-M$ to $P+M$. The vector $\underline{c}$ needs to be chosen such that

$$[c[-P] \cdots c[0] \cdots c[P]][\underline{h}_{-d} \cdots \underline{h}_0 \cdots \underline{h}_d] = [0 \cdots 1 \cdots 0].$$

Existence of the ZF-LE for a particular d depends on the solvability of the above set of linear equations. Assuming the matrix above has full rank, a ZF-LE can be obtained for $1 \leq d \leq P$.

5. In a DFE, let the precursor be $C(z) = \sum_{i=-P}^{0} c[i]z^{-i}$ and the postcursor $D(z) = \sum_{i=1}^{N} d[i]z^{-i}$. Assuming correct slicer decisions, the input to the slicer $y[k]$ can be written as

$$y[k] = s[k] * (h[k] * c[k] - d[k]) + n[k] * c[k],$$

where $*$ denotes convolution. Typically, $D(z)$ is chosen to cancel a part of $H(z)C(z)$. Since

$$p[i] = h[k] * c[k]|_i = \sum_{k=-P}^{0} c[k]h[i-k],$$

we let $d[i] = p[i]$ for $1 \leq i \leq N$. Assuming this choice for $D(z)$, the postcursor has been derived in terms of the precursor $C(z)$ and the channel $H(z)$. The precursor can now be determined to satisfy any criterion. In a ZF-DFE, we use (1) with $c[i] = 0$ for $1 \leq i \leq P$ to solve for a suitable $C(z)$. The ISI terms to be cancelled with the precursor have to be chosen carefully. The postcursor will then be calculated to cancel M terms in the resultant causal ISI.

6. Use a modifed version of the solution to Problem 5. Choose the postcursor in terms of the precursor and use the MMSE-LE formulation to find $C(z) = \sum_{i=-P}^{0} c[i]z^{-i}$.

7. For finding the precursor, use the solution to Problem 3 with the indices for the matrices $\underline{\alpha}$ and ϕ running only from P down to 0. Use Problem 5 for finding the postcursor in terms of the precursor.