Fundamentals of Noise

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Noise in resistors

- Random voltage fluctuations across a resistor
- Mean square value in a frequency range Δf proportional to R and T
- Independent of the material, size and shape of the conductor
- Contains equal (mean square) amplitudes at all frequencies (upto a very high frequency) $\Rightarrow$ Power contained in a frequency range Δf is the same at all frequencies or the power spectral density is a constant.

Questions

Why do these fluctuations occur?

- Electrons have thermal energy $\Rightarrow$ a finite velocity
- At random points in time, they experience collisions with lattice ions $\Rightarrow$ velocity changes
- Velocity fluctuations lead to current fluctuations

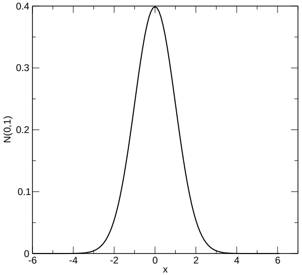
What does the power spectral density of a random signal mean?

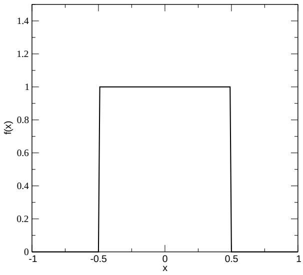
Background on random variables

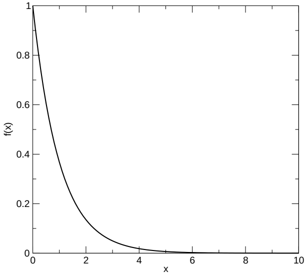
- A sample space of outcomes that occur with certain probability
- A function that maps elements of the sample space onto the real line - random variable
- Discrete random variables - example is the outcome of a coin tossing experiment
- Continuous random variables - Voltage across a resistor

Continuous Random variables

Continuous random variables are characterized by probability density functions (pdfs). Examples are







- Gaussian
- Uniform
- Exponential

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Moments of random variables

Mean

$$\mu_x = E\{X\} = \int_{-\infty}^{\infty} x f(x) dx$$

Variance

$$\sigma^2 = E\{X^2\} - \mu_x^2$$

Correlation

$$R_{XY} = E\{XY\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x, y) dx dy$$

Covariance

$$K_{XY} = E\{(X - \mu_X)(Y - \mu_Y)\}$$

- If $K_{XY} = 0$, X and Y are uncorrelated. The correlation coefficient is $c = \frac{K_{XY}}{\sigma_X \sigma_Y}$.

Random Processes

- Random variable as a function of time
- The value at any point in time determined by the pdf at that time
- The pdf and hence the moments could be a function of time
- If $\{X_1, \ldots ..X_n\}$ and $\{X_1 + h, \ldots .X_n + h\}$ have the same joint distributions for all $t_1, \ldots t_n$ and $h > 0$, it is n^{th} order stationary
- We will work with wide sense stationary (WSS) processes - The mean is constant and the autocorrelation, $R_x(t, t + \tau) = R_x(\tau)$

Spectrum

Finite energy signals - eg. $y(t) = e^{-at}$. The Fourier transform exists and energy spectral density is $|Y(f)|^2$

Finite power signals - eg. $y(t) = \sin(2\pi f_o t) + \sin(6\pi f_o t)$. In this case, can find the spectrum of the average power - eg. for $y(t)$ it is $\frac{1}{2}$ at frequencies f_o and $3f_o$.

Random signals are finite power signals, but are not periodic and the Fourier transform does not exist. The question is how do we find the frequency distribution of average power?

Limit the extent of the signal

$$x_T(t) = x(t), \quad -T/2 \le t \le T/2$$

Find the Fourier transform $X_T(f)$ of this signal. The energy spectral density is defined as

$$ESD_T = E\{|X_T(f)|^2\}$$

The power spectral density (PSD) is defined as

$$S_x(f) = \lim_{T\to\infty} \frac{E\{|X_T(f)|^2\}}{T}$$

The total power in the signal is

$$P_x(f) = \int_{-\infty}^{\infty} S_x(f)df$$

Weiner-Khinchin Theorem

The inverse Fourier transform of $|X_T(f)|^2$ is $x_T(t) * x_T(-t)$ i.e.

$$\mathcal{F}^{-1}\left[\frac{E\{X_T(f)|^2\}}{T}\right] = \frac{1}{T}\int_{-T/2}^{T/2} E\{x_T(t)x_T(t+\tau)\}dt$$

But for WSS signals,

$$R_X(\tau) = E\{x_T(t)x_T(t+\tau)\}$$

Therefore,

$$\mathcal{F}\left[R_x(\tau)\right] = S_x(f)$$

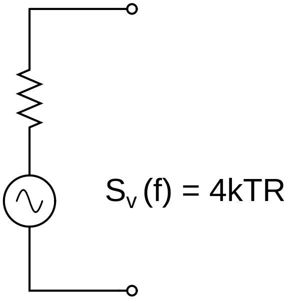
Noise in resistors

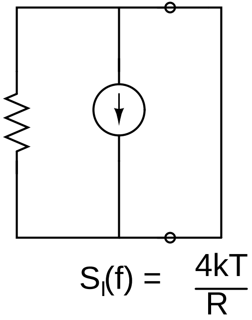
Simple model

- Collisions occur randomly - Poisson process
- Velocities before and after a collision are uncorrelated
- Average energy per degree of freedom is $\frac{kT}{2}$

With these assumptions it is possible to show that $S_I(f) = \frac{4kT}{R} \Rightarrow$ the autocorrelation is a delta function (white noise). Also, using central limit theorem, it has a Gaussian distribution. However, continuous time white noise has infinite power and is therefore an idealization.

Representation of a noisy resistor





Noise in Linear time-invariant systems

If h(t) is the impulse response of an LTI system,

$$y(t) = h(t) * x(t)$$

Assume $x(t)$ is a WSS process with autocorrelation $R_x(\tau)$

$$\begin{aligned} E\{y(t)y(t+\tau)\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)E\{x(t-\alpha)x(t+\tau-\beta)\}d\alpha d\beta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_x(\tau+\beta-\alpha)d\alpha d\beta \end{aligned}$$

$y(t)$ is also a WSS process and

$$R_y(\tau) = h(\tau) * h(-\tau) * R_x(\tau)$$

PSD and Variance

If $H(f) = \mathcal{F}\{h(t)\}$, the power spectral density at the output is

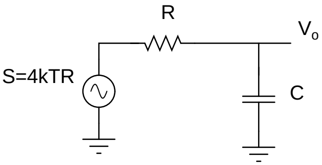
$$S_y(f) = \mathcal{F}\{R_y(\tau)\} = |H(f)|^2 S_x(f)$$

Noise power at the output is

$$R_y(0) = \int_{-\infty}^{\infty} S_y(f) df$$

In linear systems, if the input has a Gaussian distribution, the output is also Gaussian

Example - RC Circuit



- The output PSD is

$$S_y(f) = \frac{4kTR}{1 + 4\pi^2 f^2 R^2 C^2}$$

- The total power at the output is

$$\begin{aligned} P_y &= \int_0^\infty \frac{4kTR}{1 + 4\pi^2 f^2 R^2 C^2} df \\ &= \frac{kT}{C} \end{aligned}$$

Linear two ports - RLC circuits

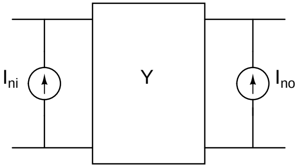
- Resistors are the noise sources
- Assuming N noise sources, the in terms of the y parameters, the output current can be written as

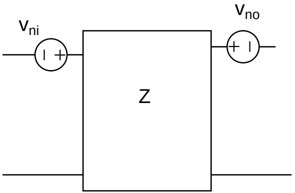
$$
\begin{aligned}
I_{in} &= y_{ii}V_{in} + y_{io}V_o + \sum_{j=1}^{N} y_{ij}V_j \\
I_{out} &= y_{oi}V_{in} + y_{oo}V_o + \sum_{j=1}^{N} y_{oj}V_j
\end{aligned}
$$

- V_j is the random noise source due to the j^{th} resistor

Noisy RLC two ports - Representation

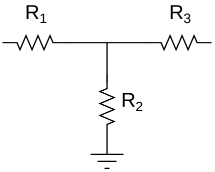
If $I_{ni} = -\sum_{j=1}^{N} y_{ij} V_j$ and $I_{no} = -\sum_{j=1}^{N} y_{oj} V_j$, the two-port can be represented as a noiseless network with two additional noise current at the two ports. Can also have a Z network, with noise voltage sources at the two ports

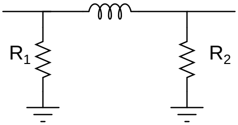




To get I_{ni} and I_{no}, find the short circuit current at the two ports. For v_{no} and v_{ni}, find the open circuit voltage

Examples





$$v_{ni}^2 = 4kT(R_1 + R_2)\Delta f$$
$$v_{no}^2 = 4kT(R_3 + R_2)\Delta f$$

$$I_{ni}^2 = \frac{4kT}{R_1}\Delta f$$

$$I_{no}^2 = \frac{4kT}{R_2}\Delta f$$

- The two noise sources are correlated

The two sources are uncorrelated

Generalized Nyquist theorem

Supposing we have an RLC network and we wish to find the noise at the output port. First remove all noise sources.

- Connect a unit current source I_o at the output
- Voltage across resistor R_j due to I_o is $V_j = Z_{jo}$
- Power absorbed in R_j is $P_j = \frac{|Z_{jo}|^2}{R_j} \Rightarrow$ Total power dissipated in the network is

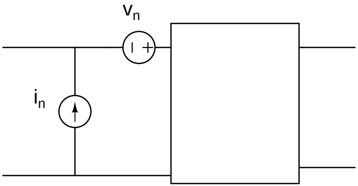
$$P = \sum_{j=1}^{N} \frac{|Z_{jo}|^2}{R_j}$$

- Power delivered to the network by the source is

$$P = Re(V_o I_o^*) = Re(Z_o)$$

- Power delivered = Power dissipated
  $\Rightarrow$
$$Re(Z_o) = \sum_{j=1}^{N} \frac{|Z_{jo}|^2}{R_j}$$
- Now include noise current sources due to resistors. Output voltage due to noise generated by resistors is
$$\bar{V_o}^2 = 4kT\Delta f \sum_{j=1}^{N} \frac{|Z_{oj}|^2}{R_j}$$
- Since the network is reciprocal $Z_{oj} = Z_{jo}$
$$\bar{V_o}^2 = 4kTRe(Z_o)\Delta f$$
  Called the Generalized Nyquist theorem

$V_n - I_n$ Representation



- Replace all noise sources in the network by an equivalent noise voltage and current source in the input port, so that the correct output noise spectral density is obtained
- Once again, the two sources could be correlated

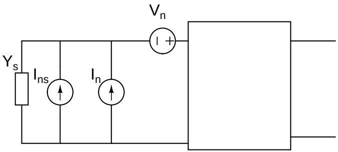
- Comparing with the $I_n - I_n$ representation,

$$
\begin{aligned}
V_n &= \frac{-I_{no}}{y_{oi}} \\
I_n &= I_{ni} - \frac{y_{ii}}{y_{oi}} I_{no}
\end{aligned}
$$

- The correlation coefficient is

$$c_{iv}(f) = \frac{E\{I_n V_n^*\}}{[E\{I_n^2\}E\{V_n^2\}]^{\frac{1}{2}}} = \frac{S_{iv}(f)}{[S_{ni}(f)S_{nv}(f)]^{\frac{1}{2}}}$$

Noise figure of a two-port network



$$F = \frac{\text{Total noise power at the output per unit bandwidth}}{\text{Output noise power per unit bandwidth due to the input}}$$

$$\begin{aligned} F &= \frac{E\{|I_{ns} + I_n + Y_s V_n|^2\}}{E\{|I_{ns}|^2\}} \\ &= 1 + \frac{S_{ni}(f)}{S_s(f)} + |Y_s|^2 \frac{S_{nv}(f)|^2}{S_s(f)} + 2Re(c_{iv} Y_s^*) \frac{[S_{ni}(f) S_{nv}(f)]^{\frac{1}{2}}}{S_s(f)} \end{aligned}$$

Depending on the correlation coefficient, one can use the right type of source impedence to minimize noise figure

References

1. H.A.Haus et al, “Representation of noise in linear twoports”, Proc. IRE, vol.48, pp 69-74, 1960
2. H.T.Friss, “Noise figure of radio receivers”, Proc. IRE, vol.32, 1944, pp 4-9-423