Modified nodal analysis

EE2015: Electrical Circuits and Networks

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18 August 2017

Generalized nodal analysis with extra variables for systematically writing down nodal equations

- Accommodates all components
  - R, L, C, M
  - Independent current and voltage sources
  - Controlled sources (VCVS, VCCS, CCVS, CCCS)
- More variables than in nodal analysis
  - Node voltages
  - Currents through voltage sources (independent and controlled)
  - Currents through inductors
  - Controlling currents of current controlled sources

Modified nodal analysis—Form of equations

$$[\mathbf{G}]\,\mathbf{v} = \mathbf{I_s}$$

- $[\mathbf{G}]$: Conductance matrix
  - Not all entries necessarily conductances
- $\mathbf{v}$: Variable vector
  - Node voltages
  - Currents through voltage sources
  - Currents through inductors
  - Controlling currents of current controlled sources
- $\mathbf{I_s}$: Vector of independent sources
  - Current and voltage sources

Every circuit element contributes to $[\mathbf{G}]$ or $\mathbf{I_s}$; Independent voltage source contributes to both.

Modified nodal analysis—variable vector

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_N \\ i_{V,1} \\ \vdots \\ i_{V,P} \\ i_{c,1} \\ \vdots \\ i_{c,Q} \end{bmatrix} \begin{array}{l} \text{Node voltages} \\ \\ \\ \text{Voltage source currents} \\ \\ \\ \text{Controlling currents} \\ \\ \\ \end{array}$$

- First label the elements and form the variable vector

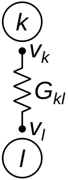
	v_1	$\cdots$	v_N	$i_{V,1}$	$\cdots$	$i_{V,P}$	$i_{c,1}$	$\cdots$	$i_{c,Q}$
node 1									
$\vdots$									
node N									
Voltage source 1									
$\vdots$									
Voltage source P									
Contr. current 1									
$\vdots$									
Contr. current Q									

- First label the elements and form the variable vector

Element contributions: "Stamps"

- Each element contributes to certain rows and columns of [**G**] or certain rows of **v**
- Element stamp indicates the rows and columns and corresponding contributions
- For each element, add the stamp to the appropriate entry of [**G**] or **v**

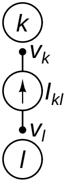
MNA stamp: Resistor



$$\begin{array}{c} \\ \text{node } k \\ \text{node } l \end{array} \begin{array}{c} \begin{array}{cc} v_k & v_l \end{array} \\ \begin{bmatrix} G_{kl} & -G_{kl} \\ -G_{kl} & G_{kl} \end{bmatrix} \end{array} \equiv \begin{array}{c} \text{node } k \\ \\ \text{node } l \end{array} \begin{array}{c} \begin{array}{ccccc} & v_k & & v_l & \end{array} \\ \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & G_{kl} & \cdots & -G_{kl} & \cdots \\ & \vdots & & \vdots & \\ \cdots & -G_{kl} & \cdots & G_{kl} & \cdots \\ & \vdots & & \vdots & \end{bmatrix} \end{array}$$

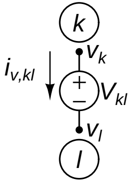
- Contributes to [**G**]

MNA stamp: Current source



$$\begin{matrix}\text{node } k\\ \text{node } l\end{matrix}\begin{bmatrix} I_{kl} \\ -I_{kl}\end{bmatrix} \equiv \begin{matrix} \\ \text{node } k \\ \\ \text{node } l \\ \\ \end{matrix}\begin{bmatrix} \vdots \\ I_{kl} \\ \vdots \\ -I_{kl} \\ \vdots \end{bmatrix}$$

- Contributes to $\mathbf{I_s}$

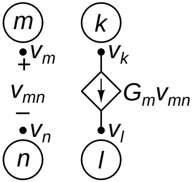


$$[\mathbf{G}] : \begin{array}{c} \\ \text{node } k \\ \text{node } l \\ i_{v,kl} \end{array} \begin{array}{c} \begin{array}{ccc} v_k & v_l & i_{v,kl} \end{array} \\ \begin{bmatrix} 0 & 0 & +1 \\ 0 & 0 & -1 \\ +1 & -1 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{I_s} : \begin{array}{c} \text{node } k \\ \text{node } l \\ i_{v,kl} \end{array} \begin{bmatrix} 0 \\ 0 \\ V_s \end{bmatrix}$$

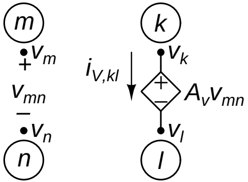
- Contributes to [**G**] and $\mathbf{I_s}$

MNA stamp: VCCS



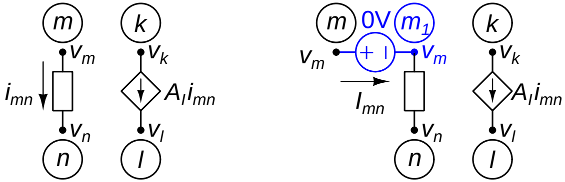
$$[\mathbf{G}] : \quad \begin{matrix} \\ \text{node } k \\ \text{node } l \end{matrix} \begin{matrix} v_m & v_n \\ \left[\begin{matrix} G_m & -G_m \\ -G_m & G_m \end{matrix}\right] \end{matrix}$$

- Contributes to [**G**]
- Note the asymmetry (rows k, l, columns m, n)



$$[\mathbf{G}]: \begin{array}{c} \\ \text{node } k \\ \text{node } l \\ i_{v,kl} \end{array} \begin{array}{c} \begin{array}{ccccc} v_k & v_l & v_m & v_n & i_{v,kl} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & -1 \\ +1 & -1 & -A & +A & 0 \end{bmatrix} \end{array}$$

- Contributes to [**G**]

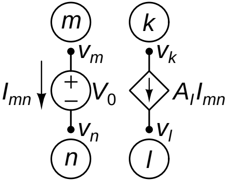


[**G**] :

	v_k	v_l	v_m	v_{m1}	i_{mn}
node k	0	0	0	0	$+A_I$
node l	0	0	0	0	$-A_I$
node m	0	0	0	0	$+1$
node m_1	0	0	0	0	-1
i_{mn}	0	0	$+1$	-1	0

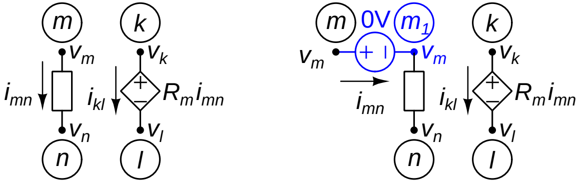
- Add a 0 V source in series with the current-sensing branch
- Extra node m_1
- Row i_{mn} corresponds to the 0 V voltage source equation

MNA stamp: CCCS (current through a voltage source)

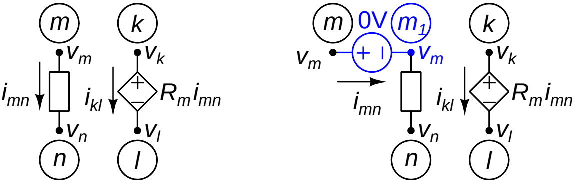


$$[\mathbf{G}] : \begin{array}{c} \\ \text{node } k \\ \text{node } l \\ \text{node } m \\ \text{node } n \\ i_{mn} \end{array} \begin{array}{c} \begin{array}{ccccc} v_k & v_l & v_m & v_n & i_{mn} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & +A_I \\ 0 & 0 & 0 & 0 & -A_I \\ 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & -1 & 0 \end{bmatrix} \end{array}$$

- No need for extra voltage source
- Rows for i_{mn} and nodes m, n: 0 V voltage source stamp



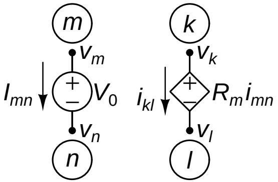
$$
[\mathbf{G}] : \quad
\begin{array}{c}
\\ \text{node } k \\ \text{node } l \\ \text{node } m \\ \text{node } m_1 \\ i_{mn} \\ i_{kl}
\end{array}
\begin{array}{c}
\begin{array}{cccccc} v_k & v_l & v_m & v_{m1} & i_{mn} & i_{kl} \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & +1 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & +1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & +1 & -1 & 0 & 0 \\
+1 & -1 & 0 & 0 & -R_m & 0
\end{bmatrix}
\end{array}
$$



[**G**] :

	v_k	v_l	v_m	v_{m1}	i_{mn}	i_{kl}
node k	0	0	0	0	0	$+1$
node l	0	0	0	0	0	-1
node m	0	0	0	0	$+1$	0
node m_1	0	0	0	0	-1	0
i_{mn}	0	0	$+1$	-1	0	0
i_{kl}	$+1$	-1	0	0	$-R_m$	0

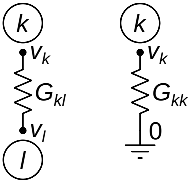
- Add a 0 V source in series with the current-sensing branch
- Extra node m_1
- Row i_{mn} corresponds to the 0 V voltage source equation
- Row i_{kl} corresponds to the CCVS equation



$$[\mathbf{G}] : \begin{array}{c} \\ \text{node } k \\ \text{node } l \\ \text{node } m \\ \text{node } n \\ i_{mn} \\ i_{kl} \end{array} \begin{array}{c} \begin{array}{cccccc} v_k & v_l & v_m & v_{m1} & i_{mn} & i_{kl} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 \\ +1 & -1 & 0 & 0 & -R_m & 0 \end{bmatrix} \end{array}$$

- No need for extra voltage source
- Rows for i_{mn} and nodes m, n: 0 V voltage source stamp

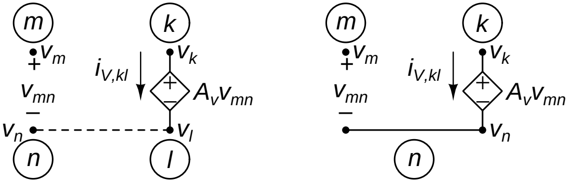
MNA: Additional remarks



$$\begin{array}{c} \\ \text{node } k \\ \text{node } l \end{array} \begin{array}{c} \begin{array}{cc} v_k & v_l \end{array} \\ \begin{bmatrix} G_{kl} & -G_{kl} \\ -G_{kl} & G_{kl} \end{bmatrix} \end{array} \equiv \text{node } k \begin{array}{c} v_k \\ [G_{kk}] \end{array}$$

- If one of the nodes k, l, m, n is the reference node, the corresponding row/column will not be present

MNA: Additional remarks



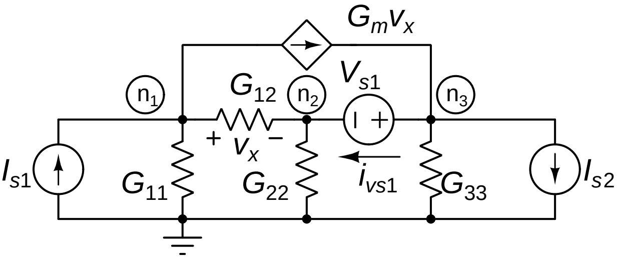
$$
[\mathbf{G}] : \quad
\begin{array}{c} \\ \text{node } k \\ \text{node } l \\ i_{v,kl} \end{array}
\begin{array}{ccccc} v_k & v_l & v_m & v_n & i_{v,kl} \\ \end{array}
\begin{bmatrix} 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & -1 \\ +1 & -1 & -A & +A & 0 \end{bmatrix}
$$

$$
\begin{array}{c} \\ \text{node } k \\ \text{node } n \\ i_{v,kl} \end{array}
\begin{array}{cccc} v_k & v_m & v_n & i_{v,kl} \\ \end{array}
\begin{bmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & -1 \\ +1 & -A & +A-1 & 0 \end{bmatrix}
$$

- If two or more nodes are the same, the corresponding terms add up

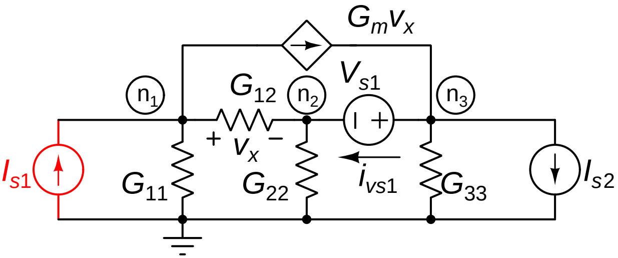
MNA: procedure

- Add 0 V voltage sources for controlling-current branches if necessary
- Assign variables for voltage source currents
- Form the variable vector $\mathbf{v}$
- Place each element's stamp appropriately in $[\mathbf{G}]$ and $\mathbf{I_s}$
- For each element of $[\mathbf{G}]$ and $\mathbf{I_s}$, add all the stamp contributions
- Solve $\mathbf{v} = [\mathbf{G}]^{-1}\,\mathbf{I_s}$

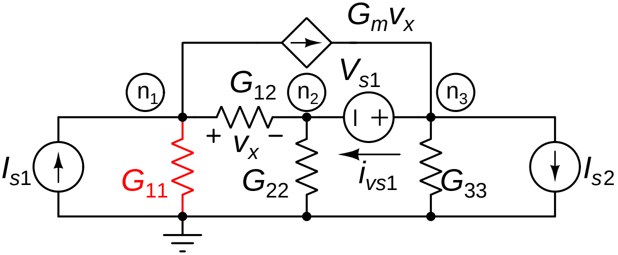


$$[\mathbf{G}]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ i_{vs1} \end{bmatrix} = \mathbf{I_s}$$

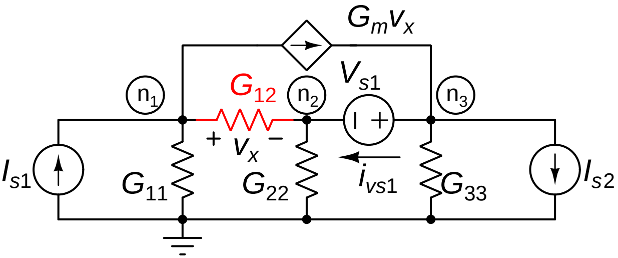
- 4 node circuit; 3 node voltages
- Variable vector: $[v_1\ v_2\ v_3\ i_{vs1}]^T$
- One extra variable: i_{vs1}
- $[\mathbf{G}]$: 4×4 matrix; $\mathbf{I_s}$: 4 element vector



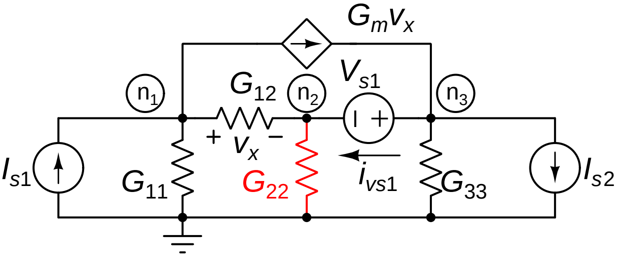
$$\mathbf{I_s} : \begin{bmatrix} I_{s1} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



$$[\mathbf{G}] : \begin{bmatrix} G_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

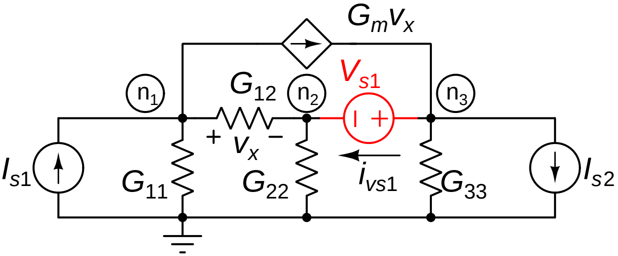


$$[\mathbf{G}] : \begin{bmatrix} G_{12} & -G_{12} & 0 & 0 \\ -G_{12} & G_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

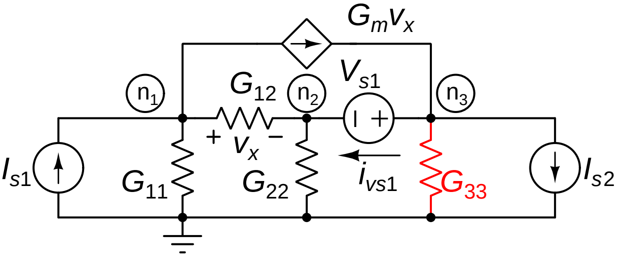


$$[\mathbf{G}] : \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & G_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

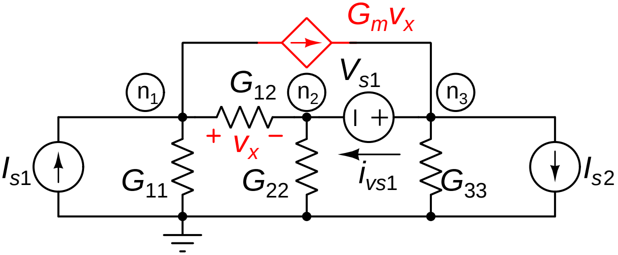
MNA example 1: Elements



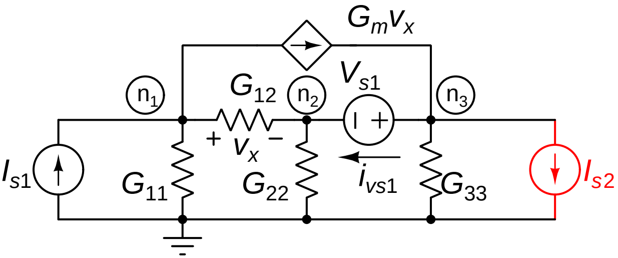
$$[\mathbf{G}]:\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & +1\\ 0 & -1 & +1 & 0\end{bmatrix}\quad \mathbf{I_s}:\begin{bmatrix}0\\ 0\\ 0\\ V_{s1}\end{bmatrix}$$



$$[\mathbf{G}] : \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & G_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

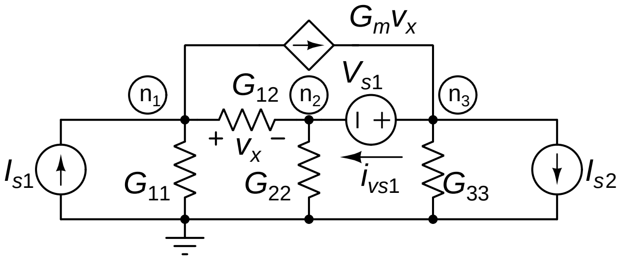


$$[\mathbf{G}] : \begin{bmatrix} G_m & -G_m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -G_m & G_m & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



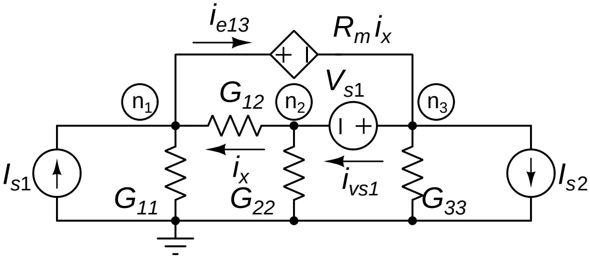
$$\mathbf{I_s} : \begin{bmatrix} 0 \\ 0 \\ -I_{s2} \\ 0 \end{bmatrix}$$

MNA example 1: Complete set of equations

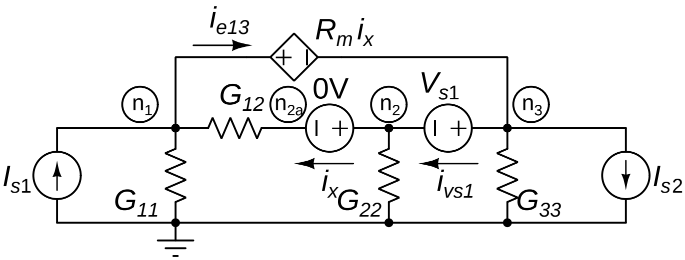


$$\begin{bmatrix} G_{11}+G_{12}+G_m & -G_{12}-G_m & 0 & 0 \\ -G_{12} & G_{12}+G_{22} & 0 & -1 \\ -G_m & G_m & G_{33} & +1 \\ 0 & -1 & +1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ i_{vs1} \end{bmatrix} = \begin{bmatrix} I_{s1} \\ 0 \\ -I_{s2} \\ V_{s1} \end{bmatrix}$$

- Solve to get $[v_1\ v_2\ v_3\ i_{vs1}]^T$
- Find all branch v, i from KCL, KVL, and element relationships

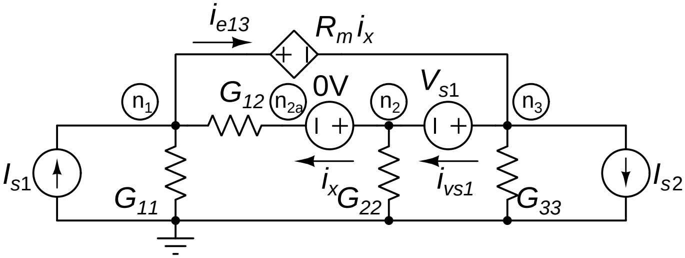


- 0 V voltage source has to be added to the controlling branch



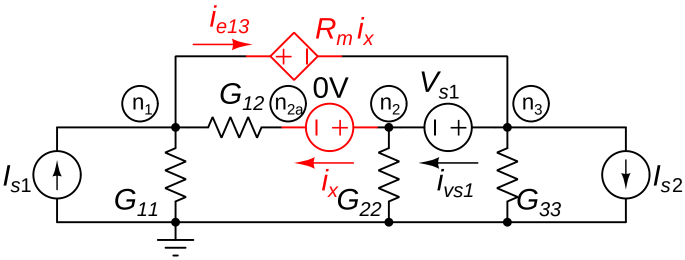
$$[\mathbf{G}] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ i_{vs1} \\ i_{e13} \\ i_x \end{bmatrix} = \mathbf{I_s}$$

- 4 node circuit; 3 node voltages
- Variable vector: $[v_1 \;\; v_2 \;\; v_3 \;\; i_{vs1}\, i_{e13}\, i_x]^T$
- One extra variable: i_{vs1}
- $[\mathbf{G}]$: 4×4 matrix; $\mathbf{I_s}$: 4 element vector



$$[\mathbf{G}]\begin{bmatrix} v_1 \\ v_2 \\ v_{2a} \\ v_3 \\ i_{vs1} \\ i_{e13} \\ i_x \end{bmatrix} = \mathbf{I_s}$$

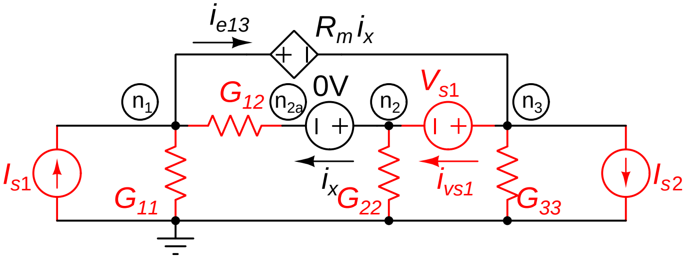
- Originally a 4 node circuit; 3 node voltages; 4 extra variables



$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & -1 & 0 & 0 & -R_m \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_{2a} \\ v_3 \\ i_{vs1} \\ i_{e13} \\ i_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- [**G**] due to the CCVS

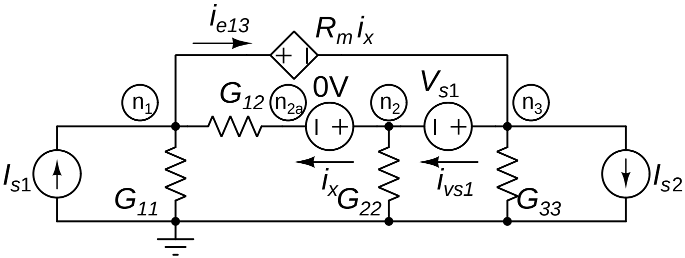
MNA example 2: Remaining elements



$$\begin{bmatrix} G_{11}+G_{12} & 0 & -G_{12} & 0 & 0 & 0 & 0 \\ 0 & G_{22} & 0 & 0 & -1 & 0 & 0 \\ -G_{12} & 0 & G_{12} & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & -1 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_{2a} \\ v_3 \\ i_{vs1} \\ i_{e13} \\ i_x \end{bmatrix} = \begin{bmatrix} I_{s1} \\ 0 \\ 0 \\ -I_{s2} \\ V_{s1} \\ 0 \\ 0 \end{bmatrix}$$

- Same as the earlier example, without G_m

MNA example 2: Complete equation setup



$$\begin{bmatrix} G_{11}+G_{12} & 0 & -G_{12} & 0 & 0 & +1 & 0 \\ 0 & G_{22} & 0 & 0 & -1 & 0 & +1 \\ -G_{12} & 0 & G_{12} & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & +1 & +1 & 0 \\ 0 & -1 & 0 & +1 & 0 & 0 & 0 \\ +1 & 0 & 0 & -1 & 0 & 0 & -R_m \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_{2a} \\ v_3 \\ i_{vs1} \\ i_{e13} \\ i_x \end{bmatrix} = \begin{bmatrix} I_{s1} \\ 0 \\ 0 \\ -I_{s2} \\ V_{s1} \\ 0 \\ 0 \end{bmatrix}$$