Circuit Analysis Using Fourier and Laplace Transforms
EE2015: Electrical Circuits and Networks

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Based on

- $\exp(st)$ being an eigenvector of linear systems
  - Steady-state response to $\exp(st)$ is $H(s) \exp(st)$ where $H(s)$ is some scaling factor
- Signals being representable as a sum (integral) of exponentials $\exp(st)$
Fourier series

Periodic $x(t)$ can be represented as sums of complex exponentials

- $x(t)$ periodic with period $T_0$
- Fundamental (radian) frequency $\omega_0 = 2\pi / T_0$

\[
x(t) = \sum_{k=-\infty}^{\infty} a_k \exp(jk\omega_0 t)\]

- $x(t)$ as a weighted sum of orthogonal basis vectors $\exp(jk\omega_0 t)$
- Fundamental frequency $\omega_0$ and its harmonics
- $a_k$: Strength of $k^{th}$ harmonic

- Coefficients $a_k$ can be derived using the relationship

\[
a_k = \frac{1}{T_0} \int_{0}^{T_0} x(t) \exp(-jk\omega_0 t) dt\]

- “Inner product” of $x(t)$ with $\exp(jk\omega_0 t)$
Fourier series

- Alternative form

\[ x(t) = a_0 + \sum_{k=1}^{\infty} b_k \cos(k\omega_0 t) + c_k \sin(k\omega_0 t) \]

- Coefficients \( b_k \) and \( c_k \) can be derived using the relationship

\[ b_k = \frac{2}{T_0} \int_0^{T_0} x(t) \cos(k\omega_0 t) \, dt \]
\[ c_k = \frac{2}{T_0} \int_0^{T_0} x(t) \sin(k\omega_0 t) \, dt \]

- Another alternative form

\[ x(t) = a_0 + \sum_{k=1}^{\infty} d_k \cos(k\omega_0 t + \phi_k) \]

- Coefficients \( b_k \) and \( c_k \) can be derived using the relationship

\[ d_k = \sqrt{b_k^2 + c_k^2} \]
\[ \phi_k = -\tan^{-1}\left( \frac{c_k}{b_k} \right) \]
If \( x(t) \) satisfies the following (Dirichlet) conditions, it can be represented by a Fourier series:

- \( x(t) \) must be absolutely integrable over a period

\[
\int_0^{T_0} |x(t)| \, dt \text{ must exist}
\]

- \( x(t) \) must have a finite number of maxima and minima in the interval \([0, T_0]\)

- \( x(t) \) must have a finite number of discontinuities in the interval \([0, T_0]\)
Aperiodic $x(t)$ can be expressed as an integral of complex exponentials

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_\omega(\omega) \exp(j\omega t) d\omega$$

- $x(t)$ as a weighted sum (integral) of orthogonal vectors $\exp(j\omega t)$
- Continuous set of frequencies $\omega$
- $X_\omega(\omega)d\omega$: Strength of the component $\exp(j\omega t)$
- $X_\omega(\omega)$: Fourier transform of $x(t)$

$X_\omega(\omega)$ can be derived using the relationship

$$X_\omega(\omega) = \int_{-\infty}^{\infty} x(t) \exp(-j\omega t) dt$$

- “Inner product” of $x(t)$ with $\exp(j\omega t)$
If \( x(t) \) satisfies either of the following conditions, it can be represented by a Fourier transform

- **Finite \( L_1 \) norm**
  \[
  \int_{-\infty}^{\infty} |x(t)| \, dt < \infty
  \]

- **Finite \( L_2 \) norm**
  \[
  \int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty
  \]

Many common signals such as sinusoids and unit step fail these criteria

- Fourier transform contains impulse functions
- Laplace transform more convenient
- $x(t)$ in volts $\Rightarrow X_\omega(\omega)$ has dimensions of volts/frequency
- $X_\omega(\omega)$: Density over frequency
- Traditionally, Fourier transform $X_f(f)$ defined as density per “Hz” (cyclic frequency)
- Scaling factor of $1/2\pi$ when integrated over $\omega$ (radian frequency)

\[
x(t) = \int_{-\infty}^{\infty} X_f(f) \exp(j2\pi ft) df
= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_\omega(\omega) \exp(j\omega t) d\omega
\]

- $X_\omega(\omega) = X_f(\omega/2\pi)$
- $X_f(f)$: volts/Hz (density per Hz) if $x(t)$ is a voltage signal

\[
X_f(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft) dt
\]

\[
X_\omega(\omega) = \int_{-\infty}^{\infty} x(t) \exp(-j\omega t) dt
\]
Fourier transform as a function of $j\omega$

- If $j\omega$ is used as the independent variable

$$x(t) = \frac{1}{j2\pi} \int_{-j\infty}^{j\infty} X(j\omega) \exp(j\omega t) d(j\omega)$$

- $X(j\omega) = X_\omega(\omega)$

- Same function, but $j\omega$ is the independent variable

- Scaling factor of $1/j2\pi$

- With $j\omega$ as the independent variable, the definition is the same as that of the Laplace transform
Fourier transform pairs

- Signals in $-\infty \leq t \leq \infty$

\[
\begin{align*}
1 & \leftrightarrow 2\pi \delta(\omega) \\
\exp(j\omega_0 t) & \leftrightarrow 2\pi \delta(\omega - \omega_0) \\
\cos(\omega_0 t) & \leftrightarrow \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0) \\
\sin(\omega_0 t) & \leftrightarrow \frac{\pi}{j} \delta(\omega - \omega_0) - \frac{\pi}{j} \delta(\omega + \omega_0) \\
\exp(-a|t|) & \leftrightarrow \frac{2a}{a^2 + \omega^2}
\end{align*}
\]

- Not very useful in circuit analysis

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Circuit Analysis Using Fourier and Laplace Transforms
Fourier transform pairs

Signals in $0 \leq t \leq \infty$

\[ u(t) \leftrightarrow \pi \delta(\omega) + \frac{1}{j\omega} \]

\[ \exp(j\omega_0 t)u(t) \leftrightarrow \pi \delta(\omega - \omega_0) + \frac{1}{j(\omega - \omega_0)} \]

\[ \cos(\omega_0 t)u(t) \leftrightarrow \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0) + \frac{j\omega}{\omega_0^2 - \omega^2} \]

\[ \sin(\omega_0 t)u(t) \leftrightarrow \frac{\pi}{j} \delta(\omega - \omega_0) - \frac{\pi}{j} \delta(\omega + \omega_0) + \frac{\omega_0}{\omega_0^2 - \omega^2} \]

\[ \exp(-at)u(t) \leftrightarrow \frac{1}{j\omega + a} \]

Useful for analyzing circuits with inputs starting at $t = 0$
Circuit analysis using the Fourier transform

- For an input $\exp(j\omega t)$, steady state output is $H(j\omega) \exp(j\omega t)$
- A general input $x(t)$ can be represented as a sum (integral) of complex exponentials $\exp(j\omega t)$ with weights $X(j\omega)d\omega/2\pi$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \exp(j\omega t) d\omega$$

- Linearity $\Rightarrow$ steady-state output $y(t)$ is the superposition of responses $H(j\omega) \exp(j\omega t)$ with the same weights $X(j\omega)d\omega/2\pi$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\omega) H(j\omega) \exp(j\omega t) d\omega$$

- Therefore, $y(t)$ is the inverse Fourier transform of $Y(j\omega) = H(j\omega)X(j\omega)$
Circuit analysis using the Fourier transform

- Calculate $X(j\omega)$
- Calculate $H(j\omega)$
  - Directly from circuit analysis
  - From differential equation, if given
- Calculate (look up) the inverse Fourier transform of $H(j\omega)X(j\omega)$ to get $y(t)$
In steady state with an input of \(\exp(j\omega t)\), “Ohms law” also valid for \(L, C\)

\[
\begin{align*}
\begin{array}{c}
\text{Resistor} \\
\text{Inductor} \\
\text{Capacitor}
\end{array}
\begin{array}{c}
\begin{array}{c|c|c|c|c}
\text{Component} & \text{Equation} & \text{\(v(t)\)} & \text{\(i(t)\)} & \text{\(v(t)/i(t)\)} \\
\hline
\text{Resistor} & v_R = R i_R & R I_R \exp(j\omega t) & I_R \exp(j\omega t) & R \\
\text{Inductor} & v_L = L \left(\frac{di_L}{dt}\right) & j \omega I_L \exp(j\omega t) & I_L \exp(j\omega t) & j \omega L \\
\text{Capacitor} & i_C = C \left(\frac{dv_C}{dt}\right) & V_C \exp(j\omega t) & j \omega V_C \exp(j\omega t) & 1 / (j \omega C)
\end{array}
\end{array}
\end{align*}
\]

- \(I_R, I_L, V_C\): Phasors corresponding to \(i_R, i_L, v_C\)
- Use analysis methods for resistive circuits with dc sources to determine \(H(j\omega)\) as ratio of currents or voltages
  - e.g. Nodal analysis, Mesh analysis, etc.
- No need to derive the differential equation
Example: Calculating the transfer function

Mesh analysis with currents $I_0$, $I_2$

\[
\begin{bmatrix}
R + \frac{1}{j\omega C_1} \\
- \frac{1}{j\omega C_1}
\end{bmatrix}
\begin{bmatrix}
I_0 \\
I_2
\end{bmatrix} = 
\begin{bmatrix}
I_0 \\
V_s
\end{bmatrix}
\]

\[
l_0 (j\omega) = \frac{(j\omega)^3 C_1 C_3 L_2 + (j\omega) (C_3 + C_1)}{(j\omega)^3 C_1 C_3 L_2 + (j\omega)^2 C_3 L_2 + (j\omega) (C_3 + C_1) R + 1}
\]

\[
V_s (j\omega) = \frac{(j\omega)^3 C_1 C_3 L_2 + (j\omega)^2 C_3 L_2 + (j\omega) (C_3 + C_1) R + 1}
\]

\[
l_2 (j\omega) = \frac{(j\omega) C_3}{(j\omega)^3 C_1 C_3 L_2 + (j\omega)^2 C_3 L_2 + (j\omega) (C_3 + C_1) R + 1}
\]

\[
V_1 = (l_0 - l_2) / (j\omega C_1), \quad V_3 = l_2 / (j\omega C_3)
\]
Example: Calculating the response of a circuit

\[ v_i(t) = V_p \exp(-at)u(t) \]

- From direct time-domain analysis, with zero initial condition

\[
\begin{align*}
\text{Steady-state response:} & \quad v_o(t) = \frac{V_p}{1 - aCR} \exp(-at)u(t) \\
\text{Transient response:} & \quad \frac{V_p}{1 - aCR} \exp(-t/RC)u(t)
\end{align*}
\]
Example: Calculating the response of a circuit

\[ v_i(t) = V_p \exp(-at)u(t) \]

Using Fourier transforms and transfer function

\[ V_o(j\omega) = \frac{V_p}{a + j\omega} \frac{1}{1 + j\omega CR} \]

\[ = \frac{V_p}{1 - aCR} \frac{1}{a + j\omega} - \frac{V_p}{1 - aCR} \frac{CR}{1 + j\omega CR} \]

From the inverse Fourier transform

\[ v_o(t) = \underbrace{\frac{V_p}{1 - aCR} \exp(-at)u(t)}_{\text{Steady-state response}} - \underbrace{\frac{V_p}{1 - aCR} \exp(-t/RC)u(t)}_{\text{Transient response}} \]

We get both steady-state and transient responses with zero initial condition.
Fourier transform of the input signal

\[ v_i(t) = e^{\exp(-t)}u(t); \ V_i(j\omega) = 1/(1 + j\omega) \]

- Fourier transform magnitude and phase \((V_p = 1, \ a = 1)\)
- Shown for \(-20 \leq \omega \leq 20\)
Fourier transform of the input signal

\[ x(t) = \frac{1}{2\pi} \int_{-20}^{20} V_i(j\omega) \exp(j\omega t) d\omega \]

- Fourier transform components \( V_i(j\omega) d\omega \cdot \exp(j\omega t) \): Sinusoids from \( t = -\infty \) to \( \infty \)
  - A small number of sample sinusoids shown above
- The integral is close, but not exactly equal to \( x(t) \)
- Extending the frequency range improves the representation
How do we get the total response by summing up steady-state responses?

- Fourier transform components $V_i(j\omega)d\omega \cdot \exp(j\omega t)$: Sinusoids from $t = -\infty$ to $\infty$
- For any $t > -\infty$, the output is the **steady-state** response $H(j\omega)V_i(j\omega)d\omega \cdot \exp(j\omega t)$
- Sum (integral) of Fourier transform components produces the input $x(t)$ (e.g. $\exp(-at)u(t)$) which starts from $t = 0$
- Sum (integral) of **steady-state** responses produces the output including the response to changes at $t = 0$, i.e. including the **transient response**
- Inverse Fourier transform of $V_i(j\omega)H(j\omega)$ is the **total** zero-state response
Accommodating initial conditions

\[ \nu_C(0^-) = V_0 \]
\[ \nu_C'(0^-) = 0 \]
\[ i_L(0^-) = I_0 \]
\[ i_L'(0^-) = 0 \]

- A capacitor cannot be distinguished from a capacitor in series with a constant voltage source
- An inductor cannot be distinguished from an inductor in parallel with a constant current source
- Initial conditions reduced to zero by inserting sources equal to initial conditions
- Treat initial conditions as extra step inputs and find the solution
  - Step inputs because they start at \( t = 0 \) and are constant afterwards
Accommodating initial conditions

\[ v_i(t) = V_p \exp(-at)u(t) \]

\[ v_x(t) = V_0 u(t) \]

\[ v_C(0^-) = V_0 \]

\[ v_C'(0^-) = 0 \]

\[ v_o(t) = \frac{H(j\omega)}{1 + j\omega CR} + V_x(j\omega) \frac{j\omega CR}{1 + j\omega CR} \]

\[ H(j\omega) = \frac{1}{1 + j\omega CR} + V_0 \left( \pi \delta(\omega) + \frac{1}{j\omega} \right) \frac{j\omega CR}{1 + j\omega CR} \]

\[ V_0(j\omega) = \frac{V_p}{a + j\omega} \frac{1}{1 + j\omega CR} + V_0 \left( \pi \delta(\omega) + \frac{1}{j\omega} \right) \frac{j\omega CR}{1 + j\omega CR} \]

\[ V_0(j\omega) = \frac{V_p}{1 - aCR} \left( \frac{1}{a + j\omega} - \frac{CR}{1 + j\omega CR} \right) + V_0 \frac{CR}{1 + j\omega CR} \]

\[ v_o(t) = \frac{V_p}{1 - aCR} \exp(-at)u(t) + \left( V_0 - \frac{V_p}{1 - aCR} \right) \exp(-t/RC)u(t) \]

- Impulse vanishes because \( \delta(\omega)H_x(j\omega) = \delta(\omega)H_x(0) \), and \( H_x(0) = 0 \)
Contains impulses for some commonly used signals with infinite energy
- e.g. $u(t)$, $\cos(\omega_0 t)u(t)$
- Even more problematic for signals like the ramp—Contains impulse derivative

→ Laplace transform eliminates these problems
Laplace transform

- Problem with Fourier transform of \(x(t)\) (zero for \(t < 0\))
  \[
  \int_{0-}^{\infty} x(t) \exp(-j\omega t) dt \text{ may not converge}
  \]

- Multiply \(x(t)\) by \(\exp(-\sigma t)\) to turn it into a finite energy signal\(^1\)

- Fourier transform of \(x(t) \exp(-\sigma t)\)
  \[
  X_{\sigma,j\omega}(j\omega) = \int_{0-}^{\infty} x(t) \exp(-\sigma t) \exp(-j\omega t) dt
  \]

- Inverse Fourier transform of \(X_{\sigma,j\omega}(j\omega)\) yields \(x(t) \exp(-\sigma t)\)
  \[
  x(t) \exp(-\sigma t) = \frac{1}{j2\pi} \int_{-j\infty}^{j\infty} X_{\sigma,j\omega}(j\omega) \exp(j\omega t) d(j\omega)
  \]

- To get \(x(t)\), multiply by \(\exp(\sigma t)\)
  \[
  x(t) = \frac{1}{j2\pi} \int_{-j\infty}^{j\infty} X_{\sigma,j\omega}(j\omega) \exp(\sigma t) \exp(j\omega t) d(j\omega)
  \]

\(^1\) Allowable values of \(\sigma\) will be clear later
Laplace transform

- Defining \( s = \sigma + j\omega \)

\[
x(t) = \frac{1}{j2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) \exp(st) \, ds
\]

- \( s \): complex variable
- Integral carried out on a line parallel to imaginary axis on the \( s \)-plane

- Representation of \( x(t) \) as a weighted sum of \( \exp(st) \) where \( s = \sigma + j\omega \)
  - \( s \) was purely imaginary in case of the Fourier transform

- Well defined weighting function \( X(s) \) for a suitable choice of \( \sigma \)

- \( X(s) \) (same as \( X_{\sigma,j\omega}(j\omega) \) with \( s = \sigma + j\omega \)) given by

\[
X(s) = \int_{0^-}^{\infty} x(t) \exp(-st) \, dt
\]

- This is the Laplace transform of \( x(t) \)
- Same definition as the Fourier transform expressed as a function of \( j\omega \)
e.g. $x(t) = u(t)$

$$\int_{0^-}^{\infty} x(t) \exp(-j\omega t) dt \text{ does not converge}$$

$$\int_{0^-}^{\infty} x(t) \exp(-st) dt \text{ converges to } \frac{1}{s} \text{ for } \sigma > 0$$

If $\sigma$ is such that Fourier transform of $x(t) \exp(-\sigma t)$ converges, $x(t)$ can be written as sum (integral) of complex exponentials with that $\sigma$

$$x(t) = \frac{1}{j2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) \exp(st) ds$$

Steady-state response to $\exp(st)$ is $H(s) \exp(st)$, so proceed as with Fourier transform
Circuit analysis using the Laplace transform

- For an input \( \exp(st) \), steady state output is \( H(s) \exp(st) \)
- A general input \( x(t) \) represented as a sum (integral)² of complex exponentials \( \exp(st) \) with weights \( X(s)ds/j2\pi \)

\[
x(t) = \frac{1}{j2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) \exp(st)ds
\]

- By linearity, steady-state \( y(t) \) is the superposition of responses \( H(s) \exp(st) \) with weights \( X(s)ds/j2\pi \)

\[
y(t) = \frac{1}{j2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} \underbrace{X(s)H(s)} \exp(st)ds
\]

- Therefore, \( y(t) \) is the inverse Laplace transform of \( Y(s) = H(s)X(s) \)

² \( \sigma \) is some value with which \( X(s) \) can be found; Value not relevant to circuit analysis as long as it exists.
Circuit analysis using the Laplace transform

\[ x(t) \rightarrow \text{Laplace transform} \rightarrow X(s) \rightarrow H(s) \rightarrow Y(s) = H(s)X(s) \rightarrow \text{Inverse Laplace transform} \rightarrow y(t) \]

- Calculate \( X(s) \)
- Calculate \( H(s) \)
  - Directly from circuit analysis
  - From differential equation, if given
- Calculate (look up) the inverse Laplace transform of \( H(s)X(s) \) to get \( y(t) \)
Signals in $0 \leq t \leq \infty$

\[
\begin{align*}
    u(t) & \iff \frac{1}{s} \\
    tu(t) & \iff \frac{1}{s^2} \\
    \exp(j\omega_0 t)u(t) & \iff \frac{1}{s - j\omega_0} \\
    \cos(\omega_0 t)u(t) & \iff \frac{s}{s^2 + \omega_0^2} \\
    \sin(\omega_0 t)u(t) & \iff \frac{\omega_0}{s^2 + \omega_0^2} \\
    \exp(-at)u(t) & \iff \frac{1}{s + a} \\
    t \exp(-at)u(t) & \iff \frac{1}{(s + a)^2}
\end{align*}
\]
In steady-state with \( \exp(st) \) input, “Ohms law” also valid for L, C

\[
\begin{align*}
R & \quad i_R \\
C & \quad i_C \\
L & \quad i_L
\end{align*}
\]

<table>
<thead>
<tr>
<th></th>
<th>( v_R = Ri_R )</th>
<th>( v(t) )</th>
<th>( i(t) )</th>
<th>( v(t)/i(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistor</td>
<td>( R )</td>
<td>( R )</td>
<td>( R )</td>
<td>( R )</td>
</tr>
<tr>
<td>Inductor</td>
<td>( v_L = L(di_L/dt) )</td>
<td>( sLI_L )</td>
<td>( sL )</td>
<td>( sL )</td>
</tr>
<tr>
<td>Capacitor</td>
<td>( i_C = C(dv_C/dt) )</td>
<td>( sCV_C )</td>
<td>( 1/(sC) )</td>
<td>( 1/(sC) )</td>
</tr>
</tbody>
</table>

- Use analysis methods for resistive circuits with dc sources to determine \( H(s) \) as ratio of currents or voltages
  - e.g. Nodal analysis, Mesh analysis, etc.
- No need to derive the differential equation
Example: Calculating the transfer function

Nodal analysis with voltages $V_1$, $V_2$

\[
\begin{bmatrix}
\frac{1}{R} + sC_1 + \frac{1}{sL_2} & -\frac{1}{sL_2} \\
-\frac{1}{sL_2} & \frac{1}{sL_2} + sC_3 + \frac{1}{R}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
= 
\begin{bmatrix}
I_s \\
0
\end{bmatrix}
\]

\[
\begin{align*}
\frac{V_1}{I_s} &= \frac{R}{s^3C_1C_3L_2R + s^2(C_1 + C_3) L_2 + s((C_1 + C_3)R + L_2/R) + 2} \\
\frac{V_2}{I_s} &= \frac{1}{s^3C_1C_3L_2R + s^2(C_1 + C_3) L_2 + s((C_1 + C_3)R + L_2/R) + 2}
\end{align*}
\]
Accommodating initial conditions

\[ v_C(0^-) = V_0 \]
\[ l'_C(0^-) = 0 \]
\[ l_L(0^-) = l_0 \]
\[ l'_L(0^-) = 0 \]

- Initial conditions reduced to zero; extra step inputs
- Circuit interpretation of the derivative operator

\[ \frac{dx}{dt} \leftrightarrow sX(s) - x(0^-) \]
\[ \frac{dx}{dt} \leftrightarrow s \left( X(s) - \frac{x(0^-)}{s} \right) \]

- Extra step input \( x(0^-)/s \)
Calculating the output with initial conditions

\[ v_i(t) = V_p \cos(\omega_0 t) u(t) \]

\[ v_i(t) + \]
\[ R \quad v_C(0-) = V_0 \]
\[ \]
\[ C \quad v_o(t) - \]
\[ v_i(t) + \]
\[ R \quad v_{C'}(0-) = 0 \]
\[ \]
\[ C \quad v_{C'}(t) - \]
\[ v_i(t) + \]
\[ v_x(t) + \]
\[ v_x(t) - \]
\[ v_{o}(t) - \]
\[ v_{o}(t) - \]

\[ v_{x}(t) = V_0 u(t) \]

\[ V_o(s) = V_p \frac{s}{s^2 + \omega_0^2} \left[ 1 + \frac{1}{1 + sCR} \right] + \frac{V_0}{s} \left[ \frac{sCR}{1 + sCR} \right] \]

\[ = \frac{V_p}{1 + (\omega_0 CR)^2} \frac{s + (\omega_0 CR)\omega_0}{s^2 + \omega_0^2} + \left( V_0 - \frac{V_p}{1 + (\omega_0 CR)^2} \right) \frac{CR}{1 + sCR} \]

\[ \text{Steady-state response} \]

\[ v_o(t) = \frac{V_p}{\sqrt{1 + (\omega_0 CR)^2}} \cos(\omega_0 t - \phi) u(t) + \left( V_0 - \frac{V_p}{1 + (\omega_0 CR)^2} \right) \exp(-t/RC) u(t) \]

\[ \phi = \tan^{-1} (\omega_0 CR) \]
Laplace transform: \( \exp(st) \) components and convergence

Constituent \( X(s) \exp(st) \) with \( \sigma = 0.1 \)

\[
\int_{0.1-j20}^{0.1+j20} X(s) \exp(st) ds
\]

\[ x(t) = u(t); X(s) = 1/s \]

Sum of exponentially modulated sinusoids with \( \sigma = 0.1 \) converges to the unit step
Laplace transform: \( \exp(st) \) components and convergence

Constituent \( X(s) \exp(st) \) with \( \sigma = 0.3 \)

\[
x(t) = \frac{1}{j2\pi} \int_{0.3-j20}^{0.3+j20} X(s) \exp(st) \, ds
\]

- \( x(t) = u(t); \ X(s) = 1/s \)
- Sum of exponentially modulated sinusoids with \( \sigma = 0.3 \) converges to the unit step
- Any \( \sigma \) in the region of convergence (ROC) would do
- For \( u(t) \), ROC is \( \sigma > 0 \)
Laplace transform: exp(st) components and convergence

Constituent $X(s) \exp(st)$ with $\sigma = 0$

$x(t) = u(t); X(s) = 1/s$

For $u(t)$, ROC is $\sigma > 0$

Sum of exponentially modulated sinusoids with $\sigma = 0$ does not converge to the unit step

This is the Fourier transform of $u(t)$ with $\pi \delta(\omega)$ missing

Zero dc part in the sum
Laplace transform: \( \exp(st) \) components and convergence

Constituent \( X(s) \exp(st) \) with \( \sigma = -0.1 \)

\[
x(t) = u(t); \quad X(s) = \frac{1}{s}
\]

For \( u(t) \), ROC is \( \sigma > 0 \),

- Sum of exponentially modulated sinusoids with \( \sigma = -0.1 \) does not converge to \( u(t) \), but converges of \( -u(-t) \! \)!
- Inverse Laplace transform formula uniquely defines the function only if the ROC is also specified
- Inverse Laplace transform of \( X(s) = 1/s \) with ROC of \( \sigma < 0 \) is \( -u(-t) \)!
Laplace transform: \( \exp(st) \) components and convergence

Constituent \( X(s) \exp(st) \) with \( \sigma = -0.3 \)

\[
x(t) \quad \frac{1}{j2\pi} \int_{-0.3-j20}^{-0.3+j20} X(s) \exp(st) ds
\]

- \( x(t) = u(t); \) \( X(s) = 1/s \)
- For \( u(t) \), ROC is \( \sigma > 0 \)
- Sum of exponentially modulated sinusoids with \( \sigma = -0.3 \) does not converge to \( u(t) \), but converges of \( -u(-t) \)!
- Inverse Laplace transform formula uniquely defines the function only if the ROC is also specified
- Inverse Laplace transform of \( X(s) = 1/s \) with ROC of \( \sigma < 0 \) is \( -u(-t) \)
Laplace transform $F(s)$ uniquely defines the function only if the ROC is also specified.

Inverse Laplace transform of $F(s)$ can be $f(t)u(t)$ (a right-sided or causal signal) as well as $-f(t)u(-t)$ (a left-sided or anti-causal signal) depending on the choice of $\sigma$.

Specifying causality or the ROC removes the ambiguity.

One-sided ($0 \leq t \leq \infty$) Laplace transform applies only to causal signals.
Impulse response

Laplace transform of $\delta(t)$ is 1
Transfer function $H(s)$: Laplace transform of the impulse response $h(t)$
Impulse response usually calculated from the Laplace transform
Step response

- Laplace transform of $u(t)$ is $1/s$
- $H(s)/s$: Laplace transform of the unit step response $h_u(t)$
- Step response usually calculated from the Laplace transform

$Y(s) = H(s)/s$
Circuits with R, L, C, controlled sources

- **Transfer function:** Rational polynomial in \( s \)
  - Transfer function from any voltage or current \( x(t) \) to any voltage or current \( y(t) \)
  
  \[
  H(s) = \frac{Y(s)}{X(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \ldots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \ldots + a_1 s + a_0}
  \]

- \( H(s) \) of the form \( N(s)/D(s) \) where \( N(s) \) and \( D(s) \) are polynomials in \( s \)

- **Differential equation relating \( y \) and \( x \)**
  \[
  a_N \frac{d^N y}{dt^N} + a_{N-1} \frac{d^{N-1} y}{dt^{N-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_M \frac{d^M x}{dt^M} + b_{M-1} \frac{d^{M-1} x}{dt^{M-1}} + \ldots + b_1 \frac{dx}{dt} + b_0 x
  \]

- \( D(s) \) corresponds to LHS of the differential equation
  - Highest power of \( s \) in \( D(s) \): Order of the transfer function

- \( N(s) \) corresponds to RHS of the differential equation

- **Transfer function:** Convenient way of getting the differential equation

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Transfer function: Rational polynomial

\[ H(s) = \frac{N(s)}{D(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \ldots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \ldots + a_1 s + a_0} \]

- Convenient form for finding dc gain \( b_0/a_0 \), high frequency behavior \( (b_M/a_N) s^{M-N} \)

Transfer function: Factored into first and second order polynomials

\[ H(s) = \frac{N(s)}{D(s)} = \frac{N_1(s) N_2(s) \cdots N_K(s)}{D_1(s) D_2(s) \cdots D_L(s)} \]

- \( K = M/2 \) (even \( M \)), \( K = (M+1)/2 \) (odd \( M \)); \( L = N/2 \) (even \( N \)), \( L = (N+1)/2 \) (odd \( N \))
- \( N_k(s) \): All second order (even \( M \)) or one first order and the rest second order (odd \( M \)); Similarly for \( D_l(s) \)
- Convenient for realizing as a cascade; combining different \( N_k \) and \( D_l \)
Transfer function: Factored into terms with zeros and poles

- Transfer function: zero, pole, gain form

\[ H(s) = \frac{N(s)}{D(s)} = k \frac{(s - z_1)(s - z_2) \cdots (s - z_M)}{(s - p_1)(s - p_2) \cdots (s - p_N)} \]

- Zeros \( z_k \), poles \( p_k \), gain \( k \)
- Convenient for seeing poles and zeros

- Transfer function: Alternative zero, pole, gain form\(^3\)

\[ H(s) = \frac{N(s)}{D(s)} = k_0 \frac{(1 - \frac{s}{z_1})(1 - \frac{s}{z_2}) \cdots (1 - \frac{s}{z_M})}{(1 - \frac{s}{p_1})(1 - \frac{s}{p_2}) \cdots (1 - \frac{s}{p_N})} \]

- Zeros \( z_k \), poles \( p_k \), gain \( k \)
- \( k_0 \): dc gain
- Convenient for seeing poles and zeros
- Convenient for drawing Bode plots

\(^3\)Cannot use when poles or zeros are at the origin
Transfer function: Partial fraction expansion

\[ H(s) = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \ldots + \frac{c_N}{s - p_N} \]

\[ h(t) = c_1 \exp(p_1 t) + c_2 \exp(p_2 t) + \ldots c_N \exp(p_N t) \]

- Convenient for finding the impulse response (natural response)
- Shown for distinct poles; Modified for repeated roots
- Terms for complex conjugate poles can be combined to get responses of type \( \exp(p_1 t) \cos(p_1 t + \phi) \)
Circuits with lumped R, L, C and controlled sources

- Causal, with natural responses of the type $\exp(pt)$
- Laplace transform of the impulse response converges with $\sigma$ greater than the largest real part among all the poles
- Can be used for analyzing the total response of any circuit (even unstable ones) with inputs which have well-defined Laplace transform
- Don’t have to worry about ROC while using the Laplace transform to analyze circuits with lumped R, L, C and controlled sources
Solve for the complete response including initial conditions
Determine the poles and zeros, evaluate stability
Write down the differential equation
Get the Fourier transform (when it exists without impulses) by substituting $s = j\omega$
Get the sinusoidal steady-state response
- Response to $\cos(\omega_0 t + \theta)$ is $|H(j\omega_0)|\cos(\omega_0 t + \theta + \angle H(j\omega_0))$

Not convenient for analysis of energy/power
- Have to use time domain or Fourier transform
Phasor analysis

- Only sinusoidal steady-state
- Convenient for fixed-frequency (e.g. power) or narrowband (e.g. RF) signals
- Easier to see cancellation of reactances etc., than with Laplace transform
  - Laplace transform requires finding zeros of polynomials
- Maybe easier to see other types of impedance transformation
Exact analysis can be tedious
Provides a lot of intuition
Can handle nonlinearity
For some problems frequency-domain analysis can be unwieldy whereas time-domain analysis is very easy
  e.g. steady-state response of a first order RC filter to a square wave—try using the Fourier series and transfer functions at the fundamental frequency and its harmonics!
  Response to $\sum_k a_k \exp(jk\omega_0 t)$ is $\sum_k a_k H(jk\omega_0) \exp(jk\omega_0 t)$

Practice all techniques on a large number of problems so that you can attack any problem