A Primer on the 2D Vector Finite Element Method

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I. Introduction

The finite element method (FEM) is a versatile tool for the solution of partial differential equations with specified boundary conditions and has been extensively applied to various engineering problems. Here, we focus on the use of a two-dimensional (2D) method using first order vector based elements to solve Maxwell's equations for the study of scattering problems from arbitrary geometries. Readers are encouraged to refer to these standard texts for elaborations [1][2], and to [3] to see the use of FEM in a research setting.

II. Theory

A. Maxwell

Our beloved Maxwell's equations in a source free medium are, assuming a $e^{j\omega t}$ time harmonic dependence[1];

$$\nabla \times \vec{E}(\vec{r}) = -\frac{\partial B(\vec{r})}{\partial t} = -j\omega\mu_0\mu_r(\vec{r})\vec{H}(\vec{r}), \quad \nabla \times \vec{H}(\vec{r}) = \frac{\partial D(\vec{r})}{\partial t} = j\omega\epsilon_0\epsilon_r(\vec{r})\vec{E}(\vec{r}) \tag{1}$$

Combined, these give us either of the two vector wave equations (with the explicit dependence on the position coordinate $\vec{r}$ dropped;

$$\nabla \times \left(\frac{1}{\epsilon_r}\nabla \times \vec{H}\right) = k_0^2\mu_r\vec{H}, \quad \nabla \times \left(\frac{1}{\mu_r}\nabla \times \vec{E}\right) = k_0^2\epsilon_r\vec{E} \tag{2}$$

Any arbitrary polarization can be expressed in terms of two orthogonal polarizations, where either $\vec{H}$ is entirely in the plane (we term[2] this transverse magnetic, TM), or where $\vec{E}$ is entirely in the plane (transverse electric, TE). Accordingly, we solve either the first of the wave equations (TM pol case), or the second (TE pol case).

B. Galerkin

With the wave equations now defined, we move closer towards the FEM. We should first be familiar with a broader mathematical technique known as Galerkin's method. Let us recast the first wave equation as $F_H(\vec{r}) = 0$, with $F_H(\vec{r}) = \nabla \times \left(\frac{1}{\epsilon_r(\vec{r})}\nabla \times \vec{H}(\vec{r})\right) - k_0^2\mu_r(\vec{r})\vec{H}(\vec{r})$. Galerkin's method gives that instead of insisting that the equation $F_H(\vec{r}) = 0$, $\forall \vec{r} \in \Omega$, where Ω is the computational domain, we enforce a weaker statement, that its dot product with a test function of our choice $\vec{T}(\vec{r})$, 'average' out to zero over the domain;

$$\int_\Omega \vec{T}(\vec{r}) \cdot F_H(\vec{r})\, d\Omega = 0 \tag{3}$$

In the FEM, we expand the unknown field, $\vec{H}$ or $\vec{E}$, along some convenient set of basis functions. By allowing $\vec{T}$ to cycle through the *same* basis set, we generate a set of equations which must be efficiently solved. Further, these basis functions have compact support in Ω, i.e. they are non-zero only over a small region in Ω. This property, along with the use of a local boundary condition, to be explained later, makes the set of equations to be sparse.

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[1]Note that this automatically fixes a right traveling plane wave to be $e^{j(\omega t - \vec{k}\cdot\vec{r})}$.

[2]This convention is not universal and one often finds the opposite convention in the literature.

C. Formulation

Elaborating on (3) gives, in the TM pol case;

$$\iint_{\Omega} \vec{T} \cdot \left[\nabla \times \left(\frac{1}{\epsilon_r} \nabla \times \vec{H} \right) - k_0^2 \mu_r \vec{H} \right] dS = 0 \tag{4}$$

We can simplify the first term of the integrand using the vector identity below;

$$\vec{A} \cdot (\nabla \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \nabla \cdot (\vec{A} \times \vec{B}) \quad \text{leading to,} \tag{5}$$

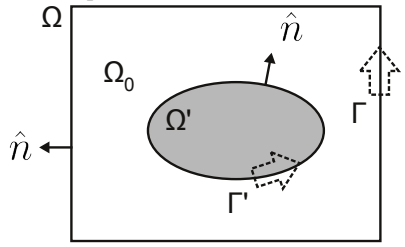
$$\vec{T} \cdot \left[\nabla \times \left(\frac{1}{\epsilon_r} \nabla \times \vec{H} \right) \right] = \left(\nabla \times \vec{T} \right) \cdot \left(\frac{1}{\epsilon_r} \nabla \times \vec{H} \right) - \nabla \cdot \left[\vec{T} \times \left(\frac{1}{\epsilon_r} \nabla \times \vec{H} \right) \right] \tag{6}$$

Substituting back and applying Green's theorem $\left[\iint_D \nabla \cdot \vec{F} \, dS = \oint_C \vec{F} \cdot \hat{n} \, dl \right]$ to the last term above gives;

$$\iint_{\Omega} \left[\left(\nabla \times \vec{T} \right) \cdot \left(\frac{1}{\epsilon_r} \nabla \times \vec{H} \right) - k_0^2 \mu_r \vec{T} \cdot \vec{H} \right] dS = \oint_{\Gamma} \vec{T} \times \left(\frac{1}{\epsilon_r} \nabla \times \vec{H} \right) \cdot \hat{n} \, dl \tag{7}$$

where Γ is the contour that encloses the domain Ω, and $\hat{n}$ is the outward normal to Γ.

Fig. 1. Computational domain



The contour integral(s) on the right is(are) the place where appropriate boundary conditions can be specified. The above relation holds true for a domain that is simply connected, and if there are missing regions withing Ω, such as from a PEC object, the contour integral on the right is modified as $\oint_{\Gamma} - \Sigma_i \oint_{\Gamma_i}$, where Γ_i is the boundary of a missing region. All contour integrals are taken in the counter-clockwise sense. In the ensuing discussion we assume there are no such regions, though it is straightforward to take them into account if needed.

D. Boundary conditions

A boundary condition specifies the behaviour of the fields on the boundaries. The outermost boundary (Γ) is non-physical, in that it doesn't exist in the physical scattering problem, but it exists because we can only admit a finite computational domain. A boundary condition must have the property that it allows as little reflection as possible of waves that are incident onto it.

The simplest boundary condition to implement on the outermost boundary Γ is the first order absorbing boundary condition, also known as the radiation boundary condition. A simple way to derive this condition is as follows. Consider an isotropic medium (ϵ_r, μ_r), and the normal incidence of a plane wave on a planar boundary, say the $x = 0$ plane, giving $E_z = e^{-jk_0 n_r x}$ as the (right traveling) incident wave. This is a fictitious boundary – (ϵ_r, μ_r) don't change across the boundary. Thus, $H_y = -\frac{1}{Z} E_z$, where $Z = Z_0 \sqrt{\mu_r / \epsilon_r} = Z_0 Z_r$, $n_r = \sqrt{\epsilon_r \mu_r}$. When E_z and H_y satisfy the above relations, there is no reflection at the boundary. Recasting the above equations gives the first order ABC, in scalar form as

$$\nabla \times H_y = \frac{jk_0 \epsilon_r}{Z_0} E_z = \frac{jk_0 \epsilon_r}{Z_0} (-Z H_y) = -jk_0 n_r H_y \tag{8}$$

and in the vector form as

$$\hat{n} \times \left(\frac{1}{\epsilon_r} \nabla \times \vec{H} \right) = -jk_0 Z_r \left(\hat{n} \times (\hat{n} \times \vec{H}) \right) \quad \text{TM pol} \tag{9}$$

$$\hat{n} \times \left(\frac{1}{\mu_r} \nabla \times \vec{E} \right) = -\frac{jk_0}{Z_r} \left(\hat{n} \times (\hat{n} \times \vec{E}) \right) \quad \text{TE pol} \tag{10}$$

It is important to note that this condition only applies to outgoing waves. In general, the fields can be decomposed into incident and scattered waves, and only the latter are required to be outgoing at every point on the outer boundary Γ. It can be easily seen that the above relations only hold when considering normal incidence on the fictitious boundary – at all other angles, the relation doesn't hold and is only an approximation.

E. Total Field Formulation

In the solution of equation (7) by the FEM, an important choice to be made is the nature of the unknown field. When the total(scattered) field is the unknown of interest, it is referred to as the total(scattered) field formulation.

In the total field formulation, we use Galerkin's equation (7) combined with the boundary conditions (9) to proceed towards the FEM implementation. First, the RHS of (7) is modified by expressing $\vec{U} = \vec{U}_i + \vec{U}_s$, where $U \in \{E, H\}$. As noted earlier, (9) is applicable only to the scattered field, $\vec{U}_s = \vec{U} - \vec{U}_i$. In the TM pol case this gives,

$$\hat{n} \times \left(\frac{1}{\epsilon_r} \nabla \times \vec{H}_s \right) = -jk_0 Z_r \left(\hat{n} \times \left(\hat{n} \times \left(\vec{H} - \vec{H}_i \right) \right) \right). \tag{11}$$

For sake of clarity, the final expression is as follows (noting that $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C}$),

$$\begin{aligned} \iint_{\Omega} \left[\left(\nabla \times \vec{T} \right) \cdot \left(\frac{1}{\epsilon_r} \nabla \times \vec{H} \right) - k_0^2 \mu_r \vec{T} \cdot \vec{H} \right] dS - \oint_{\Gamma} \left(jk_0 Z_r \right) \vec{T} \cdot \left(\hat{n} \times \left(\hat{n} \times \vec{H} \right) \right) dl = \\ \oint_{\Gamma} \vec{T} \times \left(\frac{1}{\epsilon_r} \nabla \times \vec{H}_i \right) \cdot \hat{n} \, dl - \oint_{\Gamma} \left(jk_0 Z_r \right) \vec{T} \cdot \left(\hat{n} \times \left(\hat{n} \times \vec{H}_i \right) \right) dl \end{aligned} \tag{12}$$

where the terms involving the known incident field, $\vec{H}_i$, are kept on the RHS.

F. Scattered Field Formulation

In the scattered field formulation, a domain decomposition has to be considered because the incident field satisfies the vector wave equations only in vacuum (i.e. equation (2) with $\epsilon_r = \mu_r = 1$). Consider the computational domain as shown in (1). The vacuum part of the domain, Ω_0 is bounded by contours Γ (outer) and Γ' (inner), and Ω' is occupied by a dielectric scattering object. In the vacuum domain, the equations relating $\vec{H}_s$ are straightforward,

$$\iint_{\Omega_0} \left[\left(\nabla \times \vec{T} \right) \cdot \left(\frac{1}{\epsilon_r} \nabla \times \vec{H}_s \right) - k_0^2 \mu_r \vec{T} \cdot \vec{H}_s \right] dS =$$

$$\oint_{\Gamma} \vec{T} \times \left(\frac{1}{\epsilon_r} \nabla \times \vec{H}_s \right) \cdot \hat{n} \, dl - \oint_{\Gamma'} \vec{T} \times \left(\frac{1}{\epsilon_r} \nabla \times \vec{H}_s \right) \cdot \hat{n} \, dl \quad \text{which, using (9) becomes}$$

$$\begin{aligned} \iint_{\Omega_0} \left[\left(\nabla \times \vec{T} \right) \cdot \left(\frac{1}{\epsilon_r} \nabla \times \vec{H}_s \right) - k_0^2 \mu_r \vec{T} \cdot \vec{H}_s \right] dS + \oint_{\Gamma} \left(jk_0 Z_r \right) \vec{T} \cdot \left(\hat{n} \times \left(\hat{n} \times \vec{H}_s \right) \right) dl = \\ \oint_{\Gamma'} \vec{T} \times \left(\frac{1}{\epsilon_r} \nabla \times \vec{H}_s \right) \cdot \hat{n} \, dl \end{aligned} \tag{13}$$

Note that in the above relations, $\hat{n}$ is an outward normal and $\epsilon_r = \mu_r = 1$ throughout on Γ. In case the object is immersed in a medium other than vacuum, the incident wave satisfies Maxwell's equations in *that* medium, and the ϵ_r, μ_r above take on the values of that medium.

Within the dielectric object, it makes little sense to partition the field into the incident and scattered field, and so the variable of interest is kept as the total field. We get,

$$\iint_{\Omega'} \left[\left(\nabla \times \vec{T} \right) \cdot \left(\frac{1}{\epsilon_r} \nabla \times \vec{H} \right) - k_0^2 \mu_r \vec{T} \cdot \vec{H} \right] dS = \oint_{\Gamma'} \vec{T} \times \left(\frac{1}{\epsilon_r} \nabla \times \vec{H} \right) \cdot \hat{n} \, dl \tag{14}$$

Care has to be taken on the common boundary, Γ', where the two domains share different unknowns, $\vec{H}_s$ on the Ω_0 side, and $\vec{H}$ on the Ω' side. Since $\vec{H} = \vec{H}_s + \vec{H}_i$ on Γ', the two unknowns can always be related and combining (13) and (14) above gives a consistent set of equations to solve in the scattered field formulation.

An important difference between the total (12) and scattered (13-14) formulations is the location where the incident field is introduced into the formulation. In the total-case, the incident field enters on the outermost boundary, Γ, where as in the scattered-case the incident field enters on the dielectric boundary, Γ', as evident by the RHS of (12,13,14). This has two benefits for the scattered formulation;

1) The incident field need not be propagated across the mesh from Γ to Γ', thus not experiencing field dispersion error,
2) An absorbing layer can be placed just inside Γ, which can help absorb the scattered field (in addition to the absorbing boundary condition). This is not possible in the total formulation, because then the

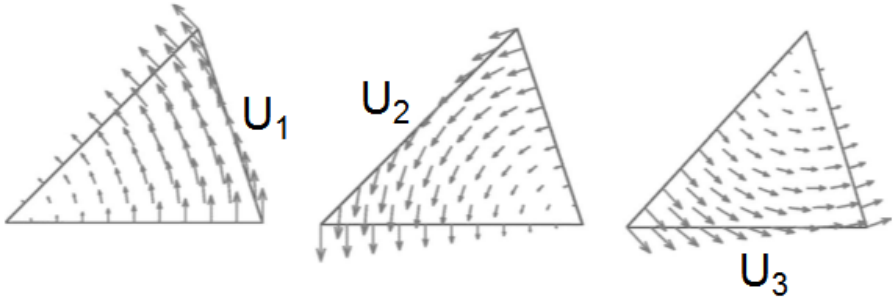
incident field would need to propagate through an absorbing layer to reach the interface Γ'. See [4] for an example of the implementation of an adiabatically absorbing layer.

III. Implementation

A. Basis functions

To make the implementation concrete, the unknown fields are first expressed in terms of vector-valued basis functions. We choose the first order Whitney basis functions. There are three basis functions per element (triangle), and these are shown graphically in Figure (2).

Fig. 2. First order Whitney elements [1]



These functions have two special properties which make them suitable for electromagnetic problems;

1) They have a constant tangential component along one edge.
2) They have no tangential components along the other two edges.

Both these properties ensure that tangential field continuity across adjacent elements is easily achieved. Thus, Maxwell's boundary conditions get satisfied for free! The curl components, however, are in general not continuous across element boundaries.

Consider an element with nodes i, j, k and edges i, j, k, where edge i is opposite nodes i, j and so on. Define the area coordinate function for an interior point P as

$$L_i = \frac{\text{Area}\,\Delta Pjk}{\text{Area}\,\Delta ijk} = \frac{(x_j y_k - x_k y_j) - x(y_k - y_j) + y(x_k - x_j)}{(x_j y_k - x_k y_j) - x_i(y_k - y_j) + y_i(x_k - x_j)} = \frac{a_i + b_i x + c_i y}{2\Delta} \tag{15}$$

This gives the basis function for edge k as

$$\vec{T}_k = l_k(L_i \nabla L_j - L_j \nabla L_i) = \frac{l_k}{4\Delta^2}(A_k + B_k y, C_k + D_k x) \tag{16}$$

With the basis functions thus defined, we express the unknown field inside an element in terms of the basis function as $\vec{U} = \Sigma_0^2 U_i \vec{T}_i$, where U_i is the constant tangential component of $\vec{T}_i$ along edge i. The objective of the FEM procedure then, is to determine the unknowns, U_i, and thus obtain the fields everywhere.

It must be noted that these functions are defined to be non-zero only inside the element – thus, these are also called 'local' basis functions. This is what leads to a sparse set of equations in the FEM; consider the product $\vec{T}_m(r) \cdot \vec{f}(\vec{T}_n(r))$, where f is some vector valued function in space; the product will be non-zero only as long as edges m, n share a common element. In other words, an edge does not couple with edges outside its immediate neighbourhood.

B. Matrix assembly

So far we have considered a single element and we now need to consider the coupling between adjacent elements in order to build the FEM system of equations.

We first note that in the previous sub-section, the orientation of the basis functions $\vec{T}$ was taken with respect to the local node and edge numbers. Now, we adopt a global edge direction convention, where an edge points from a node with a smaller global node number to a node with a higher number as seen in Figure (3). This is important, because, for example edge 5 points from node 2 to node 4, but for the same edge, element a and b have opposite (local) edge directions.

Let us perform Galerkin testing with global edge 5 of Figure (3) to demonstrate matrix assembly. This edge has two basis functions, one corresponding to each element, and both will share the same tangential

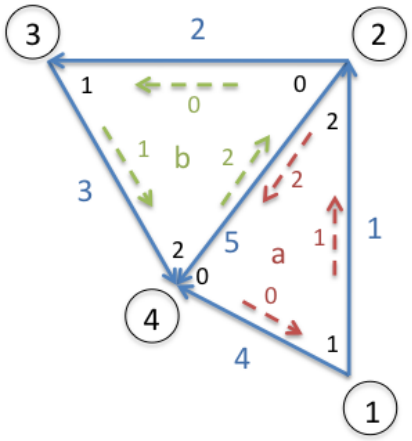


Fig. 3. Two elements a, b that share an edge. Global node numbers are in black circles, global edge numbers are marked in blue. Local node and edge numbers are marked inside each element.

component along edge 5. Consider the surface integral in equation (7), which appears in the same form in both the total and scattered formulations: $\iint_\Omega \Phi(\vec{T}, \vec{H})\, dS$, where Φ is an operator that acts as follows;

$$\Phi(\vec{T}, \vec{H}) = \left(\nabla \times \vec{T}\right) \cdot \left(\frac{1}{\epsilon_r} \nabla \times \vec{H}\right) - k_0^2 \mu_r \vec{T} \cdot \vec{H} \tag{17}$$

We have,

$$\begin{aligned} &\iint_\Omega \Phi(\vec{T}, \vec{H})\, dS = \iint_a \Phi(\vec{T}, \vec{H})\, dS + \iint_b \Phi(\vec{T}, \vec{H}),\, dS \quad \text{simplified using } \vec{H} = \Sigma U_i \vec{T}_i, \\ &= \iint_a \{U_4 \Phi(\vec{T}_5^a, \vec{T}_4^a) + U_1 \Phi(\vec{T}_5^a, \vec{T}_1^a) + U_5 \Phi(\vec{T}_5^a, \vec{T}_5^a)\} + \iint_b \{U_2 \Phi(\vec{T}_5^b, \vec{T}_2^b) + U_3 \Phi(\vec{T}_5^b, \vec{T}_3^b) + U_5 \Phi(\vec{T}_5^b, \vec{T}_5^b)\}\, dS \end{aligned} \tag{18}$$

To be clear, $\vec{T}$ can have two super-scripts, a or b, depending on which element it is non-zero over. To be clearer still, $\vec{T}_i^a$ is non-zero only over element a, but zero over element b, whereas $\vec{T}_i^b$ is the other way, i.e. non-zero only over element b, but zero over element a, even though the edge number is the same for both, i.

This above takes care of the first term on the LHS of the total (12) and scattered formulations (13,14). The second term on the LHS of (12,13) is rather simple to handle;

$$\oint \vec{T}_m \cdot (\hat{n} \times (\hat{n} \times \vec{H}))\, dl = \oint \vec{T}_m \cdot (\hat{n} \times (\hat{n} \times (\Sigma_{i=0}^3 U_{g(i)} \vec{T}_{g(i)}))\, dl = -\delta_{m,g(i)} U_m l_m \tag{19}$$

i.e. it is non-zero only when the edge m couples to itself and is on the boundary. Here, $g(i)$ is a local-to-global node number mapping function, and l_m is the length of edge m. The super-script on $\vec{T}$ has been dropped because an edge on the boundary belongs only to one element.

Finally, only if edge 5 belongs to the outer boundary Γ in the total field formulation (12), or on the dielectric boundary Γ' (see Figure 1) in the scattered field formulation (12,13) we will have a non-zero RHS term. This term, called B_5, purely involves a coupling of the incidence field with edge $\vec{T}_5$ over the boundary.

The resulting row in the matrix equation for edge 5 looks like, with $A_{m,n}^a = \iint_a \Phi(\vec{T}_m^a, \vec{T}_n^a)\, dS$

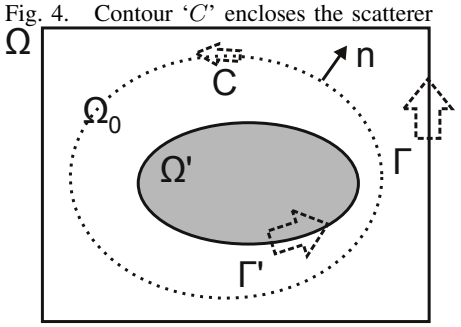
$$\begin{bmatrix} A_{5,1}^a & A_{5,2}^b & A_{5,3}^b & A_{5,4}^a & A_{5,5}^a + A_{5,5}^b \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \begin{bmatrix} B_5 \end{bmatrix} \tag{20}$$

Proceeding in the demonstrated manner, we can assemble the entire symmetric sparse matrix, $A\,u = b$, and solve for all the unknown field components U_i. The field at any point interior to element a is then simply $\Sigma_{i=0}^2 U_{g(i)} \vec{T}_{g(i)}$, and the complimentary field is proportional to the curl – which in the case of the first order elements, is constant for a given edge over the entire element ($\nabla \times \vec{T} = l(D - B)/(4\Delta^2)$), from 16).

C. Computing the far-field

Once the fields have been computing by matrix solution, we are often interested in finding either the far-field, or the radar cross-section (RCS) of the scattering object.

Fig. 4. Contour 'C' encloses the scatterer



Consider Figure (4); the dotted contour C encloses the scattering object. By applying the field equivalence principle [5], we can replace the fields to the interior of C by zero fields, and instead place surface electric and magnetic currents on C in order to maintain Maxwell's tangential boundary conditions. These currents are given by $\vec{M}(\vec{r'}) = \vec{E}(\vec{r'}) \times \hat{n}$, $\vec{J}(\vec{r'}) = \hat{n} \times \vec{H}(\vec{r'})$, where the primed coordinate $\vec{r'}$ refers to a point on the contour C. The fields exterior to C remain unchanged. It is important to note that the fields E, H in the above relations are the total fields, so care must be taken in the case of the scattered field formulation to add the incident field correctly in order to get the total field on the contour C.

The far-field can be evaluated at any point $\vec{r}$ by utilizing the far-field approximations of the 2D Green's function, and the final result is in terms of a line integral over the contour C;

$$E_z^f(\vec{r}) = \sqrt{\frac{k_0}{8\pi}} \frac{e^{-i(k_0 r - \pi/4)}}{\sqrt{r}} \times \left[\oint_C \hat{z} \cdot \left(\hat{r} \times \vec{M}(\vec{r'}) + Z_0 \mu_r \hat{r} \times \hat{r} \times \vec{J}(\vec{r'}) \right) e^{ik_0 \hat{r} \cdot \vec{r'}} dl' \right] \text{ (TM pol)} \quad (21)$$

$$H_z^f(\vec{r}) = \sqrt{\frac{k_0}{8\pi}} \frac{e^{-i(k_0 r - \pi/4)}}{\sqrt{r}} \times \left[\oint_C \hat{z} \cdot \left(-\hat{r} \times \vec{J}(\vec{r'}) + \frac{\epsilon_r}{Z_0} \hat{r} \times \hat{r} \times \vec{M}(\vec{r'}) \right) e^{ik_0 \hat{r} \cdot \vec{r'}} dl' \right] \text{ (TE pol)} \quad (22)$$

Note that this relation holds only in the case that the Green's function can be written, which is possible for homogeneous media. Thus, no part of the scatterer should be outside C.

We can also calculate the radar cross-section (RCS), σ_{2D} of the scattering object by using the following relations

$$\sigma_{2D} = \lim_{r \to \infty} 2\pi r \left| \frac{E_z^f}{E_z^i} \right|^2 \text{ (TM pol), and } \sigma_{2D} = \lim_{r \to \infty} 2\pi r \left| \frac{H_z^f}{H_z^i} \right|^2 \text{ (TE pol)} \quad (23)$$

It can be observed that the RCS is independent of the far-field distance, r.

The subscript i in the above relations refer to the (known) incident field. In the simplest form, this can be a plane wave of the form $U_i^z = \exp(-j\vec{k} \cdot \vec{r})$. This won't work in the case of scattering geometries that are more complicated, such as semi-infinite mediums; think of trying to evaluate the RCS of a ship – the surrounding ocean is an example of a semi-infinite medium which runs into some of the boundaries of the computational domain. Here, we need to 'taper' the incident field appropriately, such that the field amplitude at the corners of the semi-infinite medium is low enough to suppress the unavoidable corner diffraction problem. See [6],[3] for more details.

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