Computational Electromagnetics : Hybrid methods

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Topics in this module

1. Motivation
2. The Finite Element-Boundary Integral method

Table of Contents

What are hybrid methods, why do we need them?

e.g. of a scattering problem

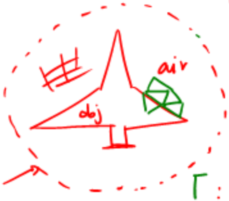
RCS ?

Volume integral: • just object ✓

• dense system of eqns ×

$O(n^3)$

recap: how to solve using FEM?



✓ Sparse system.

$O(n \log n)$

× Discretize air

× Inexact BC.

Γ: apply RBC.

What are hybrid methods, why do we need them?

recap: how to solve using IEM?

↳ Surface integral.

$V_1 : G_1$

2D TM

↳ Extinction thm to solve :

↳ Unknowns are tangential fields

$(\nabla\phi\cdot\hat{n})$, (ϕ)

$\propto \vec{H}_{tan}$ $\qquad$ E_z

$G_1 \longrightarrow$ free space ✓

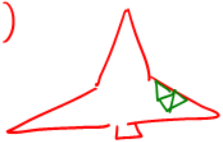
$G_2 \longrightarrow$ Hetero obj ✗

$$\nabla^2 G + k^2 \varepsilon_r(r)\, G(r,r') = -\delta(r,r')$$

Can we do better?

$\to G_1 \propto H_0^{(2)}(k|r-r'|)$

replace



↳ Use FEM for interior obj

$\Rightarrow$ Discretize.

↳ Use BI for boundary condn instead of ABC.

Table of Contents

Finite Element-Boundary Integral (FE-BI)

2D vector FEM, TM polarization

1. Maxwell's equations: $\vec{F}_H(\vec{r}) = \nabla \times [\frac{1}{\epsilon_r}\nabla \times \vec{H}] - k_0^2\mu_r\vec{H} = 0$
2. Weighted residual method: $\iint_\Omega \vec{T}_m(\vec{r}) \cdot \vec{F}_H(\vec{r})\, d\vec{r} = 0$
3. $\Longrightarrow \iint_\Omega \left[(\nabla \times \vec{T}_m) \cdot (\frac{1}{\epsilon_r}\nabla \times \vec{H}) - k_0^2\mu_r\vec{T}_m \cdot \vec{H}\right] d\vec{r} = \oint_\Gamma \vec{T}_m \times \frac{1}{\epsilon_r}(\nabla \times \vec{H}) \cdot \hat{n}\, dl$
4. ABC approximation to RHS: $\nabla \times \vec{H}_s \approx -jk\,(\hat{n} \times \vec{H}_s)$

weak form.

Exact, no B.C.

Unknowns ? = No of edges in triangulation of domain

n : no of interior edges

m : no of boundary edges

$\vec{H} = \sum_{i=1}^{3} u_i \vec{T}_i(r)$

mag of tangential fields.

⇒ Sparse sys : $(n+m) \times (n+m)$ matrix size.

Extinction thm

TM $\rightarrow$ (H_x, H_y, E_z)

FE-BI: How to combine?

v_1 g_1

v_2 g_2

Pulse basis.

2D surface integral formulation, TM polarization

5. $\oint_\Gamma [g_1(r,r')(\nabla E_z(r)\cdot\hat n) - E_z(r)\nabla g_1(r,r')\cdot\hat n]\,dl = E_z^i(r')$
6. $\oint_\Gamma [g_2(r,r')(\nabla E_z(r)\cdot\hat n) - E_z(r)\nabla g_2(r,r')\cdot\hat n]\,dl = 0$
7. Recall: $\vec H_{tan} = \frac{j}{\omega\mu_0}(\nabla E_z\cdot\hat n)\,\hat t$

$(\nabla E_z\cdot\hat n)$, (E_z) $\rightarrow$ variables

(H_{tan}, E_z) $\xrightarrow{\text{Take eqn(5)}}$ (2m unknowns. m eqns.)

Key is in boundary condition of FEM – replace ABC by BI

$$\vec T \times \frac{1}{\varepsilon_r}(\nabla\times\vec H)\cdot\hat n = \vec T\times j\omega\varepsilon_0\vec E\cdot\hat n = j\omega_0\,\underbrace{\vec T}_{\hat t}\times E_z\hat z\cdot\hat n$$

$$= \pm j\omega_0 E_z$$

$|\vec T| = 1$ along boundary

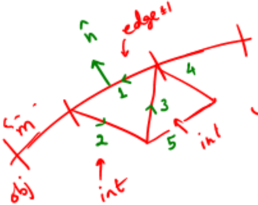
$$\oint(\ )\,dl = \pm j\omega_0 E_z\,\underbrace{\ell}_{\text{length}}$$

$v_1,\ v_2, v_3,\ v_4, v_5$

FE-BI: How to combine?

Recall shorthand: $\Phi(\vec{T}_m, \vec{H}) = \left[(\nabla \times \vec{T}_m) \cdot (\frac{1}{\epsilon_r}\nabla \times \vec{H}) - k_0^2 \mu_r \vec{T}_m \cdot \vec{H}\right]$

$E_z(r) = \sum_{i=1}^{m} v_i \underbrace{p_i(r)}_{\text{pulse basis}}$



$\vec{H}(r) = \sum_{i=1}^{3} u_i \vec{T}_i(r)$. Choose $\vec{T}_1$ as testing fn.

$$\iint_\Omega \Phi(\vec{T}_1, \vec{H})\, d\vec{r} = u_1 \underbrace{\iint \Phi(\bar{T}_1, \bar{T}_1) d\bar{r}}_{C_{pq}} + u_2 \iint \Phi(\bar{T}_1, \bar{T}_2) d\bar{r} + u_3 \iint \Phi(\bar{T}_1, \bar{T}_3)$$

$$= \pm j w_0 l_1 v_1 \quad \text{— (1)}$$

n+m eqns total from FEM.

n → do not involve v_i's.

m → involve v_i

No of unknowns:

$$\underbrace{n+m}_{\vec{H}_{tan}} + \underbrace{m}_{E_z}$$

BI: $\sum_{q \in \Gamma} A_{pq} u_q + B_{pq} v_q = e_p$ — (2)

⇒ m eqns in total.

FE-BI: Budgeting the variables & solution

① → $U_1(\,) + U_2(\,) + U_3(\,) + V_1(\,) = 0$ FEM

② → $[U_1(\,) + U_{-}(\,) + \cdots_m] + [V_1(\,) + \cdots_m] = e_1$ BI.

FEM: testing on int

FEM: testing on boundary edges

BI: only boundary edges ↳ eqn 2

$$\begin{bmatrix} \checkmark & \checkmark & 0 \\ \checkmark & \checkmark & \checkmark \\ 0 & \checkmark\checkmark & \checkmark\checkmark \end{bmatrix} \begin{bmatrix} U_I \\ U_B \\ V_B \end{bmatrix}_{(n+2m)\times 1} = \begin{bmatrix} 0 \\ 0 \\ e \end{bmatrix}$$

solve.

✓ ⇒ sparse

✓✓ ⇒ Dense.

BC → exact ✓

Air → No need ✓

Matrix → No longer sparse ✗

Schur's compliment.

$Ax=b$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad A^{-1}, D^{-1} \ldots$$

Topics that were covered in this module

1. Motivation
2. The Finite Element-Boundary Integral method

References: