Computational Electromagnetics : Finite Difference Time Domain Methods – Perfectly Matched Layers

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Topics in this module

1. Failure of Absorbing Boundary Conditions

2. The remedy via Perfectly Matched Layers

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Evanescent waves in the ABC: Consider 1D lossy medium

$\sigma \neq 0$

ct ct ct $\to e^{j\omega t}$

Recall : $\underbrace{\nabla \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) + \frac{\partial \vec{D}(\vec{r})}{\partial t}}$ $\approx (\sigma + j\omega\epsilon)\vec{E}(\vec{r}) = j\omega\epsilon_0 \underbrace{(\epsilon_r - j\frac{\sigma}{\omega\epsilon_0})}_{\epsilon_r'}\vec{E}(\vec{r})$

$\epsilon_r' \to$ complex.

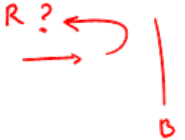
Wave Eqn: $\nabla^2 E(r) + \underbrace{k_0^2\epsilon_r'}E(r) = 0 \quad \Longrightarrow$ soln: $\quad E(r) = \exp(j(\omega t \pm k'x))$

$\pm \to$ FWD & BKWD

$k'^2 = k_0^2 \epsilon_r' \Rightarrow \boxed{k' = k_r - j k_i}$

FWD wave: $e^{j(\omega t - k'x)} = e^{j(\omega t - k_r x)} e^{-k_i x}$ decays as wave travels.

BKWD wave: $e^{j(\omega t + k'x)} = e^{j(\omega t + k_r x)} e^{k_i x}$ " " " "



lossless $\to R = 0$

ABCs fail even in 1D

↓ if medium is lossy.

BC: $\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)E = 0$

$E_{tot} = \text{inc} + \text{ref}.$

inc ↓ FWD ↓ $e^{j(\omega t - k_r x) - k_i x}$

ref. ↓ BKWD ↘ $R e^{j(\omega t + k_r x) + k_i x}$

Imposing ABC : -

$$\left[-jk_r - k_i + R(jk_r + k_i)\right] + \frac{1}{c}\left[j\omega + Rj\omega\right] = 0$$

$$R = \left[\frac{k_r - \omega/c - jk_i}{k_r + \omega/c + jk_i}\right]$$

lossless $\Rightarrow k_i = 0$

$R = 0$? $c = \omega/k_r$

lossy ? $k_i \neq 0$

R=0 Not possible.

Evanescent waves.

↳ work around → larger dist between obj & boundary.

① Non normal $\Rightarrow R \neq 0$

② lossy mediums

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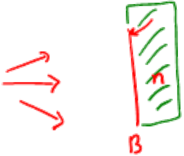
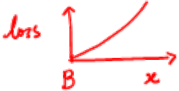
Absorbing material based PML [Berenger 1994]



1) Normal inc : $R = \frac{n-1}{n+1}$

↳ make loss increasly 'adiabatic'



Poor man's "PML".

↳ ① R = 0 for any angle
② works for evanescent waves

↓

2 interpretations of PML:

1) Absorbing material which is anisotropic. (Phy)

2) Coordinate streching method. (Math) ✓

$\nabla_e \times \bar{E} = -j\omega\mu\bar{H}$, $\nabla_h \times \bar{H} = j\omega\varepsilon\bar{E}$, $\nabla_e \cdot \varepsilon E = \rho$, $\nabla_h \cdot \mu\bar{H} = 0$

$\nabla_e = \hat{x}\frac{1}{e_x}\frac{\partial}{\partial x} + \hat{y}\frac{1}{e_y}\frac{\partial}{\partial y} + \hat{z}\frac{1}{e_z}\frac{\partial}{\partial z}$, $\nabla_h = \left(\frac{1}{h_x}\frac{\partial}{\partial x}, \frac{1}{h_y}\frac{\partial}{\partial y}, \frac{1}{h_z}\frac{\partial}{\partial z}\right)$ NEW ops.

$E = E_0 e^{\pm j\bar{k}\cdot\bar{r}}$ is a soln $\vec{k}$ needs some interpretation.

Example of using the 'stretched coordinates'

One-dim: $\vec{E} = \underbrace{e^{jk_z z}}_{E_x} \hat{x}$

wave along $\hat{z}$

$$\underline{\nabla_e} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{1}{e_x}\frac{\partial}{\partial x} & \frac{1}{e_y}\frac{\partial}{\partial y} & \frac{1}{e_z}\frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \hat{y} \frac{1}{e_z}\frac{\partial}{\partial z} E_x = j\frac{k_z}{e_z} e^{jk_z z} \hat{y} = -j\omega\mu \bar{H}$$

$$\Rightarrow H_y = -\frac{k_z}{e_z} \frac{e^{jk_z z}}{\omega\mu}$$

Plug into $\nabla_h \times \bar{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{1}{h_x}\frac{\partial}{\partial x} & \frac{1}{h_y}\frac{\partial}{\partial y} & \frac{1}{h_z}\frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix} = -\hat{x}\frac{1}{h_z}\frac{\partial H_y}{\partial z}$

$$= \frac{+j k_z^2}{h_z e_z \omega\mu} e^{jk_z z} \hat{x} = j\omega\varepsilon \vec{E} = j\omega\varepsilon e^{jk_z z} \hat{x}$$

$$\Rightarrow \frac{k_z^2}{h_z e_z} = \omega^2 \mu\varepsilon = \left(\frac{\omega}{c}\right)^2 \quad ①$$

$k_z = \sqrt{h_z e_z}\left(\frac{\omega}{c}\right)$

New dispersion relation.

Generalization to 3D

$\rightarrow (k_x, k_y, k_z) \rightarrow$ wave in 3D. $\quad \underline{e^{j\vec{k}\cdot\vec{r}}} \rightarrow$ time conv $e^{-j\omega t}$

$$\underline{k_0^2} = \left(\frac{\omega}{c}\right)^2 = \frac{k_z^2}{\underbrace{e_z h_z}} + \frac{k_y^2}{\underbrace{e_y h_y}} + \frac{k_x^2}{\underbrace{e_x h_x}} \quad \text{— (2)}$$

define: $\vec{k}_e = \left(\frac{k_x}{e_x}, \frac{k_y}{e_y}, \frac{k_z}{e_z}\right)$, $\quad \vec{k}_h = \left(\frac{k_x}{h_x}, \frac{k_y}{h_y}, \frac{k_z}{h_z}\right) \Rightarrow \left(\frac{\omega}{c}\right)^2 = \vec{k}_e \cdot \vec{k}_h$ ✓

$\nabla_e \times \vec{E} \rightarrow \vec{k}_e \times \vec{E} = -\omega\mu\bar{H}$

$\nabla_h \times \vec{H} \rightarrow \vec{k}_h \times \vec{H} = \omega\varepsilon\bar{E}$

(2) $\Rightarrow$ Eqn of an ellipsoid.

a) Set $e_x = h_x$, $e_y = h_y$, $e_z = h_z$

– called a "Matched" medium.

soln to (2).

$$k_x = k_0\sqrt{e_x h_x}\ \sin\theta\cos\phi$$

$$k_y = k_0\sqrt{e_y h_y}\ \sin\theta\sin\phi$$

$$k_z = \underbrace{k_0\sqrt{e_z h_z}}\ \cos\theta$$

Matched

$$\underline{k_z = k_0 e_z \cos\theta}$$

b) If we make e_z to be complex

$e_z = p + jq$

$\rightarrow e^{jk_z z} \rightarrow e^{jk_0 e_z \cos\theta z}$

$\rightarrow e^{jk_0 p\cos\theta z}\, e^{-k_0 q\cos\theta z}$

Evanescent wave!

coordinate stretching

$\Leftrightarrow$ Generating Evanescent waves

Waves at an interface: phase matching

plane waves:

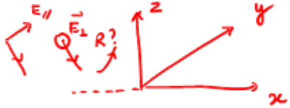
TM : E_z, H_x, H_y

✓ TE : H_z, E_x, E_y

$$\vec{E}_i = \vec{E}_o \, e^{+j\bar{k}_i\cdot\bar{r}}, \quad \text{(inc)}$$

$$\vec{E}_r = R\bar{E}_o \, e^{+j\bar{k}_r\cdot\bar{r}} \quad \text{(ref)}$$

$$\vec{E}_t = T\bar{E}_o \, e^{+j\bar{k}_t\cdot\bar{r}} \quad \text{(trans)}$$



region 1 ($z>0$)

$z=0$ (intf)

region 2 ($z<0$)

Boundary conds for tangential fields

$\because$ No z-comp $\quad \vec{E}_i + \vec{E}_r = \vec{E}_t \quad$ (at $z=0$)

$$\bar{E}_o \, e^{+j\bar{k}_i\cdot\bar{r}} + R\bar{E}_o \, e^{+j\bar{k}_r\cdot\bar{r}} = T\bar{E}_o \, e^{+j\bar{k}_t\cdot\bar{r}}$$

true at all x, y.

$$\bar{k}_i\cdot\bar{r} = k_{ix}\,x + k_{iy}\,y$$

Phase matching: $\underline{k_{ix} = k_{rx} = k_{tx}}$ & $\underline{k_{iy} = k_{ry} = k_{ty}}$

$\Rightarrow \quad 1 + R = T \quad$ (TE) — (1)

Next: $\vec{H}_{tan}$ conserved at interface..

$\nabla \times E = -\frac{\partial B}{\partial t} \rightarrow j\vec{k} \times \vec{E} = -1 \times -j\omega\mu\vec{H} \rightarrow \vec{k} \times \vec{E} = \omega\mu H$

Waves at an interface: tangential boundary conditions

Recall: $\vec{k}_e \times \vec{E} = +\omega\mu\vec{H}$, $\vec{k}_h \times \vec{H} = -\omega\epsilon\vec{E}$, $\vec{k}_{ie} = \left(\frac{k_{ix}}{e_x}, \frac{k_{iy}}{e_y}, \frac{k_{iz}}{e_z}\right)$, $\vec{k}_{re}$, $\vec{k}_{te}$ similarly.

Reg 1: inc + ref $= \frac{\vec{k}_{ie} \times \vec{E}_0}{\omega\mu_1} e^{j\bar{k}_i \cdot \bar{r}} + R\,\frac{\vec{k}_{re} \times \vec{E}_0}{\omega\mu_1} e^{j\bar{k}_r \cdot \bar{r}}$ (inc + ref)

Reg 2: trans: $= \frac{\vec{k}_{te} \times \bar{E}_0}{\omega\mu_2} e^{j\bar{k}_t \cdot \bar{r}}$

$\vec{H}_{tan,1} = \vec{H}_{tan,2}$ $\qquad k_{1z} e_{2z} \mu_2 [1 - R] = T k_{2z} e_{1z} \mu_1$ ②

$$R^{TE} = \frac{k_{1z} e_{2z} \mu_2 - k_{2z} e_{1z} \mu_1}{k_{1z} e_{2z} \mu_2 + k_{2z} e_{1z} \mu_1}$$

$\omega/c = k$

Waves at an interface: tangential boundary conditions

Define: $k_{1z} = k_{iz}, \quad k_{2z} = k_{tz}, \quad k_{rz} = -k_{1z}$

$1 \to$ Region 1

$2 \to$ Region 2.

TM pol: $$R = \frac{k_{1z} h_{2z} \varepsilon_2 - k_{2z} h_{1z} \varepsilon_1}{k_{1z} h_{2z} \varepsilon_2 + k_{2z} h_{1z} \varepsilon_1}.$$

Goal: Make R as small as possible.

Num: $$\underbrace{\left(\omega\sqrt{\mu_1\varepsilon_1}\sqrt{e_{1z}h_{1z}}\cos\theta_1\right)}^{k_{1z}} h_{2z}\varepsilon_2 - \left(\omega\sqrt{\mu_2\varepsilon_2}\sqrt{e_{2z}h_{2z}}\cos\theta_2\right) h_{1z}\varepsilon_1$$

$$\propto (e_{1z}e_{2z} - e_{2z}e_{1z}) \qquad \text{TM}$$

$$= 0$$

$k_0 = \omega\sqrt{\mu_0\varepsilon_0}$

$[e_x = h_x,\ e_y = h_y,\ e_z = h_z]$

Perfectly matched interface

Recall: $k_x = k_0\sqrt{e_x h_x}\sin\theta\cos\phi,\ \ k_y = k_0\sqrt{e_y h_y}\sin\theta\sin\phi,\ \ k_z = k_0\sqrt{e_z h_z}\cos\theta$

Region 2 → Our control.

Phase matching: ① — $k_{1x} = k_{2x}$ and $k_{1y} = k_{2y}$. ②

$$\omega\sqrt{\mu_1\varepsilon_1}\sqrt{e_{1x}h_{1x}}\sin\theta_1\cos\phi_1 = \omega\sqrt{\mu_2\varepsilon_2}\sqrt{e_{2x}h_{2x}}\sin\theta_2\cos\phi_2$$

$$\omega\sqrt{\mu_1\varepsilon_1}\sqrt{e_{1y}h_{1y}}\sin\theta_1\sin\phi_1 = \omega\sqrt{\mu_2\varepsilon_2}\sqrt{e_{2y}h_{2y}}\sin\theta_2\sin\phi_2 .$$

$\left.\begin{array}{l}\varepsilon_1 = \varepsilon_2\\ \mu_1 = \mu_2 .\end{array}\right]$ fix

$$\left.\begin{array}{l} e_{1x}\sin\theta_1\cos\phi_1 = e_{2x}\sin\theta_2\cos\phi_2 \\ e_{1y}\sin\theta_1\sin\phi_1 = e_{2y}\sin\theta_2\sin\phi_2 . \end{array}\right\}$$

say choose $e_{1x} = e_{2x}$, $e_{1y} = e_{2y}$ $\Rightarrow$ $\theta_1 = \theta_2$, $\phi_1 = \phi_2$

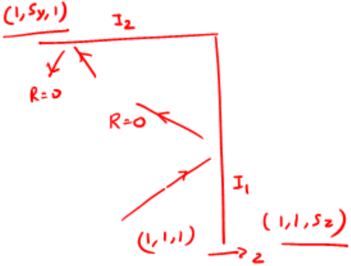
Choice of coordinate stretch parameters

Reg 1: vacuum: , $\mu = \mu_0$, $\epsilon = \epsilon_0$

$$(e_{1x}, e_{1y}, e_{1z}, h_{1x}, h_{1y}, h_{1z}) = (1, 1, 1, 1, 1, 1)$$

Reg 2 : PML

$$(e_{2x}, e_{2y}, e_{2z}, h_{2x}, h_{2y}, h_{2z}) = (1, 1, s_z, 1, 1, s_z) \checkmark$$



$\Rightarrow$ created an interface with 0 reflections

- both pols
- any inc angle

Summary of PML theory

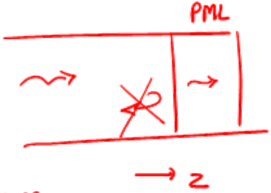
$$k_z = \overbrace{\omega\sqrt{\mu_0\epsilon_0}}^{k_0}\, s_z \cos\theta$$

$s_z \rightarrow p + jq$

$s_z \rightarrow$ In our hand.

$$e^{jk_{zz}z} = e^{jk_0 p\cos\theta z}\, e^{-k_0 q z\cos\theta}$$

$e^{-k_0 q z\cos\theta} \rightarrow$ decays.

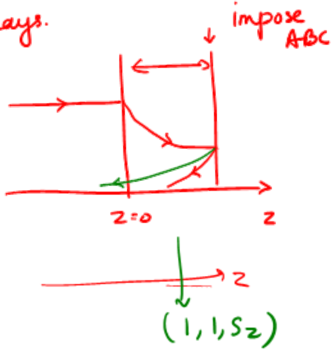


Time conv: $e^{-j\omega t}$

$\Rightarrow s = p + jq$

Time conv: $e^{+j\omega t}$

$\Rightarrow s = p - jq$



Implementing into FDTD: Comparisons with lossy media

$$(\nabla \times \vec{H}) = \hat{x}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{y}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \hat{z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$ ← (unit vecs)

$$\hat{x} \times \vec{H} = \hat{x} \times [\hat{x} H_x + \hat{y} H_y + \hat{z} H_z] = [\hat{z} H_y - \hat{y} H_z]$$

$$\nabla \times \vec{H} = \underbrace{\frac{\partial}{\partial x}(\hat{x} \times \vec{H})}_{①} + \underbrace{\frac{\partial}{\partial y}(\hat{y} \times \vec{H})}_{②} + \underbrace{\frac{\partial}{\partial z}(\hat{z} \times \vec{H})}_{③} \quad — ①$$ (partial derivatives)

Has 3 vecs.

lossy media (Ohm's Law)

Not along $\hat{x}, \hat{y}, \hat{z}$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + j\omega\varepsilon_0\varepsilon_r \vec{E}$$

$$= j\omega\varepsilon_0 \underbrace{\left[\varepsilon_r - \frac{j\sigma}{\omega\varepsilon_0}\right]}_{\varepsilon'} \vec{E} = j\omega\varepsilon_0\varepsilon' \left[\vec{E}_{s_x} + \vec{E}_{s_y} + \vec{E}_{s_z}\right] \quad — ②$$

$$\boxed{\vec{E}_{s_x} \triangleq \frac{1}{j\omega\varepsilon_0\varepsilon'}\left(\frac{\partial}{\partial x}\hat{x} \times \vec{H}\right) \quad — ③}$$

Implementing into FDTD: Setting parameters

Putting into PML.

$$\nabla_h \times \bar{H} = j\omega\epsilon_0 \bar{E}$$

↓ only 3rd comp

$$\frac{1}{s_z}\frac{\partial}{\partial z}(\hat{z}\times\bar{H}) = j\omega\epsilon_0 \bar{E}_{sz} \leftarrow \text{PML}$$

& so on for each comp.

For lossy media s_z

$$\frac{\partial}{\partial z}(\hat{z}\times\bar{H}) = j\omega\epsilon_0\left[\epsilon_r - j\frac{\sigma}{\omega\epsilon_0}\right]\bar{E}_{sz}$$

$$\Rightarrow s_z = 1 - j\frac{\sigma}{\omega\epsilon_0}, \quad s_x = 1, \quad s_y = 1$$

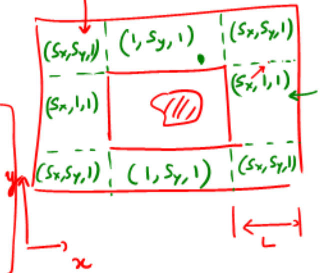
$$=$$

$$\sigma(x) = \left(\frac{x}{L}\right)^m \qquad 0 \le x \le L$$

How to implement?

$$\nabla\times H^{n-1/2} = \epsilon \dot{E}^{n-1/2} + \sigma E^{n-1/2}$$

$$= \epsilon_0\epsilon_r\left(\frac{E^n - E^{n-1}}{\Delta t}\right) + \sigma\left(\frac{E^n + E^{n-1}}{2}\right)$$



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References:

* Ch 12 of Computational Methods for Electromagnetics - Peterson, Ray, Mitra
* Computational Electrodynamics: The Finite-Difference Time-Domain Method – Allen Taflove (the 'Bible' for FDTD)
* Chew, W. C. and Weedon, W. H. (1994), A 3D perfectly matched medium from modified maxwell's equations with stretched coordinates. Microw. Opt. Technol. Lett., 7: 599-604.