Computational Electromagnetics : Method of Moments

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Topics in this module

1. Motivation
2. Linear Vector Spaces

Math

3. Formulating the Method of Moments

applications.

4. MoM: Surface Integral Equations
5. MoM: Volume Integral Equations

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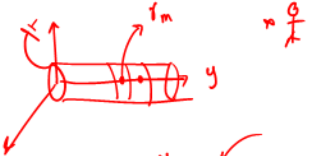

Recap of a problem already solved

$V(r_m) = \frac{1}{4\pi\varepsilon_0}\sum_{n=1}^{N}\int(\ \ )\,b_n(r')\,dr'$

$\Downarrow$

$1V$

Recall potential problem (finding $V(r)\,\forall r$)

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathsf{L}} \frac{\rho(\vec{r}')}{R}\,dl'}$$

$N\times N$

Steps: (1) Find ρ, (2) then V.

Problems with this approach? ✓ Need large 'N'.

✓ basis functions can be better

Not enforcing $V(r) = V_0$ at pts other than r_m.

Method of Moments.

Want a more robust method.

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From continuous to the discrete world

Integral/differential operators $\rightarrow$

Linear operator

we want

known

f

$\rightarrow$ $$\mathbb{L}\,\phi(r) = f(r)$$

(a) Need a basis for $D(\mathbb{L})$

b_n : basis

(b) Need a basis for $R(\mathbb{L})$

t_n : testing

Condn on f?

f must be in $\mathbb{R}(\mathbb{L})$

Once we discretize $\rightarrow$

finite dimensional vector space V $\rightarrow V_N$

$W \rightarrow W_N$

Characterized by:

$\{ b_n \ , \ n = 1, \dots, N \}$

$\left.\begin{array}{l} \text{1) Linearly indepn} \\ \text{2) Span the vector space} \end{array}\right\}$ basis

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$\mathbb{L}\,\phi(r) = f(r)$

orthonormal.

$(h(r), k(r)) = \int_a^b h(r')\,k(r')\,dr'$

Formulating a system of equations

Express $\phi(r)$ in the basis $\{b_n(r)\}_{n=1}^N$

$\phi(r) = \sum_{n=1}^{N} \phi_n\, b_n(r)$

$(b_m(r), \phi(r)) = \phi_m$

Similarly for $f(r)$ in basis $\{t_n(r)\}_{n=1}^N$

$f(r) = \sum_{n=1}^{N} f_n\, t_n(r)$ $\quad (t_n(r), f(r))$

Boundary condn? e.g. $\mathbb{L} = \frac{d}{dx}$ & $\mathbb{L}\phi(r) = 1$

unknowns now are: ϕ_n, knowns: $b_n(r), t_n(r), f_n$

$\mathbb{L} \sum_{n=1}^{N} \phi_n\, b_n(r) = \sum_{n=1}^{N} f_n t_n(r) = \sum_{n=1}^{N} \phi_n\, \mathbb{L}\, b_n(r)$

Choosing one $t_m(r)$ 'testing' fn gives:

1 eqn, N vars

$(t_m, \underset{\text{Eqn}}{\uparrow}) \rightarrow \sum_{n=1}^{N} \phi_n \underbrace{(t_m(r), \mathbb{L} b_n(r))} = f_m = (t_m(r), f(r))$

Overall matrix equation becomes: $A x = c$

$A_{mn} = (t_m(r), \mathbb{L} b_n(r))$ $\quad$ CEM action.

$c_m = f_m$

Old wine in new bottle

Choose testing/basis to be the same: Galerkin's method.

In the first problem of $\frac{1}{4\pi\epsilon_0}\int_{\mathsf{L}} \frac{\rho(\vec{r}')}{R}\, dl' = V(\vec{r})$, $\rightarrow V(r_m)$

how to describe the **old** solution procedure in the **new** language?

1) basis: $\rho(r') = \sum \rho_n \underline{b_n(r')}$

2) testing: $V(r_m) = \int_0^{\ell} v(r)\, \underline{\delta(r - r_m)}\, dr$

$t_m(r) = b_m(r)$

$\Rightarrow \underline{t_m(r) = \delta(r - r_m)}$

pulse basis, delta testing

$\downarrow$

point testing

$\downarrow$

point collaction method.

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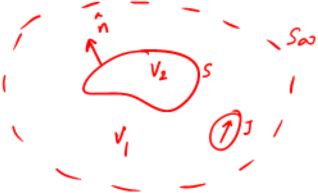
Surface Integral Equations: Recap



$$\phi_i(r') - \oint [g_1(r,r')\nabla\phi_1(r) - \phi_1(r)\nabla g_1(r,r')]\cdot\hat{n}\,dl$$

$$= \begin{cases} \phi_1(r') & r' \in V_1 \\ 0 & r' \in V_2 \end{cases}$$

Huygen's –

Extinction thm

Similarly for region 2:

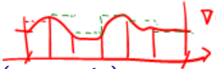
$$\oint [g_2(r,r')\nabla\phi_2(r) - \phi_2(r)\nabla g_2(r,r')]\cdot\hat{n}\,dl$$

$$= \begin{cases} \phi_2(r') & r' \in V_2 \\ 0 & r' \in V_1 \end{cases}$$

known: g_1, g_2, ϕ_i

unknown: $\phi(r)$, $\nabla\phi\cdot\hat{n}$ → a_n, b_n

✓ pulse basis $p_n(r)$ delta testing → MoM



Surface Integral Equations: Recap (contd.)

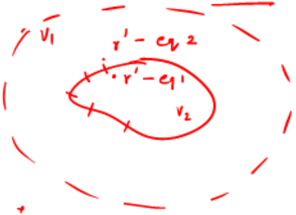
var. of intg? unprimed

Use only the Extinction theorem:

$$\oint [g_1(r,r')\nabla\phi(r)\cdot\hat{n} - \phi(r)\nabla g_1(r,r')\cdot\hat{n}]\, dl = \phi_i(r'), \quad r' \in V_2$$

$$\oint [g_2(r,r')\nabla\phi(r)\cdot\hat{n} - \phi(r)\nabla g_2(r,r')\cdot\hat{n}]\, dl = 0, \quad r' \in V_1$$

Where to choose r'?



$\nabla\phi\cdot\hat{n} = \sum_{n=1}^{N} a_n p_n(r)$, $\quad \phi(r) = \sum_{n=1}^{N} b_n p_n(r)$

$r' \in V_1$

$r' \in V_2$

Boundary Integral method

— eq(1)

Extended BC method.

$\int_{seg_i} g_1(r,r')\, dl$

$\int \nabla g_1(r,r')\hat{n}\, dl$

$\int g \, dl$ $\int \nabla g \cdot \hat{n} \, dl$

2D

Surface Integrals: Which terms are problematic?

$g(r, r') = -\frac{j}{4} H_0^{(2)}(k|r - r'|)$

Call $\rho = |r - r'| = \sqrt{(x-x')^2 + (y-y')^2}$

For $\rho \ll 1$:

$H_0^{(2)}(k\rho) \approx 1 - j\frac{2}{\pi}(\ln \frac{k\rho}{2} + \gamma)$ Euler const ≈ 0.57

Both g and ∇g blow up as $\rho \to 0$

What about ∇g? Use $\frac{dH_0^{(2)}(x)}{dx} = -H_1^{(2)}(x)$

$\nabla g = \left[\frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y}\right] = \frac{-j}{4}\left[-k \frac{2(x-x')}{2\sqrt{(\ )^2+(\ )^2}}\hat{x} + -k\frac{2(y-y')}{2\rho}\hat{y}\right] H_1^{(2)}(k\rho)$

$= \frac{jk}{4} H_1^{(2)}(k\rho)\left[\frac{(x-x')\hat{x} + (y-y')\hat{y}}{\rho}\right]$

$= \frac{jk}{4} H_1^{(2)}(k\rho)\,\hat{\mathscr{r}}$, $\vec{\mathscr{r}} = r - r'$

Thus, care while integration:

- Segments where $r \neq r'$ $\to$ Numerical quadrature rules
- Segments where $r = r'$ $\to$ Singular integrals

Surface Integrals: Kinds of singularities?

$$H_0^{(2)}(x) \approx 1 - j\frac{2}{\pi}\left(\ln(\tfrac{x}{2}) + \gamma\right) \qquad \leftarrow x \ll 1 \rightarrow \qquad H_1^{(2)}(x) \approx \frac{x}{2} + \frac{2j}{\pi}\frac{1}{x}$$

$$\lim_{\varepsilon \to 0} \int_\varepsilon^a \ln x \, dx = \Big[x \ln x - x\Big]_\varepsilon^a$$

$$= (a \ln a - a) - (\varepsilon \ln \varepsilon - \varepsilon)$$

$$\lim_{\varepsilon \to 0} \frac{\ln \varepsilon}{1/\varepsilon} \to \frac{1/\varepsilon}{-1/\varepsilon^2} = \varepsilon \to 0$$

$$= a \ln a - a \quad \text{convergent}$$

$$\Sigma\, w_k f(x_k)$$

$$\lim_{\varepsilon \to 0} \int_\varepsilon^a \frac{1}{x}\, dx = \ln x \Big|_\varepsilon^a$$

$$= \ln a - \ln \varepsilon$$

divergent.

Thus the singularity of g is integrable but not of ∇g

$\int_{-a}^{b} \ln|x| \, dx$

What happens when you integrate past a singularity?

Improper integral e.g. $\boxed{\int_{-a}^{b} \frac{1}{x} dx}$ and both $a, b > 0$. Since $\frac{1}{x} \to \infty$ as $x \to 0$,

Rewrite as: $\int_{-a}^{b} \frac{1}{x} dx =$ $\int_{-a}^{-\eta} \frac{1}{x} dx + \int_{\epsilon}^{b} \frac{1}{x} dx$

If BOTH η, ϵ approach zero independently, and the limit exists, then we say the integral is convergent. Is that true here?

$$\int_{-a}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{b} \frac{dx}{x} = (\ln \epsilon - \ln a) + (\ln b - \ln \epsilon) = \ln b/a .$$

$$\to \int_{-a}^{-2\epsilon} \frac{dx}{x} + \int_{\epsilon}^{b} \frac{dx}{x} = \ln 2\epsilon - \ln a + \ln b - \ln \epsilon = \ln b/a + \ln 2 .$$

$-a \qquad b$

$x \to -y$

No! It is divergent. But this exists $\to$ $\int_{-a}^{-\epsilon} + \int_{\epsilon}^{b}$

$\int_{-a}^{b} f(x) dx = \underbrace{PV\left(\int_{-a}^{-\epsilon} + \int_{\epsilon}^{b} \right)} + \underbrace{\int_{S_\epsilon} f(x) dx}$

Residue.

Called the Cauchy principle value (PV) of the integral.

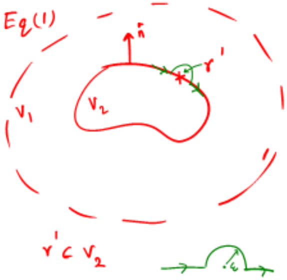
$PV \int f(x) dx \leftrightarrow ⨍ f(x) dx$

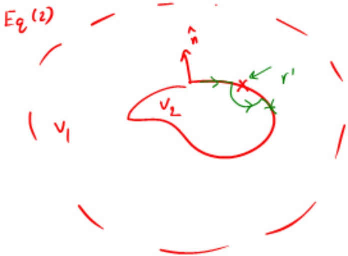
Back to the surface integral equations:

$$\oint [g_1(r,r')\nabla\phi(r)\cdot\hat{n} - \phi(r)\nabla g_1(r,r')\cdot\hat{n}]\,dl = \phi_i(r'), \quad r' \in V_2$$

$$\oint [g_2(r,r')\nabla\phi(r)\cdot\hat{n} - \phi(r)\nabla g_2(r,r')\cdot\hat{n}]\,dl = 0, \quad r' \in V_1$$

How do we change the integration contours?





$\int_0^l dl = l$

$\nabla g = \frac{jk}{4} H_1^{(2)}(k\rho)\,\hat{r}$ $\quad \frac{r - r'}{|r - r'|}$ $\quad H_1^{(2)}(x) \approx \frac{x}{2} + \frac{j}{\pi}\frac{2}{x}$

Putting it together: evaluating the integrals

1)

2)

$\int_a^b dl = \pi\varepsilon$

$\int_\pi^0 \varepsilon\, d\theta = -\pi\varepsilon$ ✗

$\int_\pi^0 \varepsilon\,(-d\theta) = \pi\varepsilon$ ✓

a)

$I_1 = \int_{S_+} \nabla g(r, r')\cdot\hat{n}\, dl = \frac{jk}{4}\int_\pi^0 H_1^{(2)}(k\rho)\,(-\varepsilon\, d\theta) = \frac{-jk\varepsilon}{4}\int_\pi^0 \left[\frac{k\varepsilon}{2} + \frac{j}{\pi}\frac{2}{k\varepsilon}\right] d\theta$

$\hat{n} = \hat{r}$ $\quad I_1 = \lim_{\varepsilon \to 0} \frac{-jk\varepsilon}{4} \times \frac{j\,2}{\pi k} \times \frac{1}{\varepsilon} \times -\pi = -\frac{1}{2}$

b)

$I_2 = \int_{S_-} \nabla g(r, r')\cdot\hat{n}\, dl = \frac{jk}{4}\int_\pi^0 H_1^{(2)}(k\rho)\,(-1)\,(-\varepsilon\, d\theta)$

$= +\frac{1}{2}.$

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1 e.g.

2nd e.g.

Volume Integral Equations: Motivation

Recap what we already know to solve:

$$\nabla^2 \phi(r) + k^2 \phi(r) = f(r) = j\omega\mu J(r)$$

$$\nabla^2 g(r,r') + k^2 g(r,r') = -\delta(r,r') \quad \text{—(2)}$$

$$\mathcal{L} = \nabla^2 + k^2$$

$\alpha = 0$

to solve this: $x^2 y'' + x y' + (x^2 - \alpha^2) y = 0$ (Bessel's Diff eqn). (3)

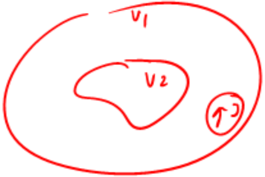
$kr = x$

$$\phi(r) = -\int g(r,r') f(r') dr'$$

const.

Homogeneous object.

Volume Integral Equations: Setting up



How is our current problem different? → $\varepsilon_r(r) = \varepsilon_r(x,y)$ within V_2

When there is no object:

$$\nabla^2 E_i(r) + k_0^2 E_i = j\omega\mu J_z(r) \quad -(1)$$

Add the object V_2

$$\nabla^2 E(r) + k_0^2 \varepsilon_r(r) E(r) = j\omega\mu \underbrace{J_z(r)}_{\text{fn of } r} \quad -(2)$$

$$\nabla^2 [E(r) - E_i(r)] + k_0^2 \underline{\varepsilon_r(r)}\, \underline{E(r)} - k_0^2 \underline{E_i(r)} = 0 \quad -(3)$$

use ?

$$\nabla^2 g(r,r') + \underline{k_0}^2 g(r,r') = -\delta(r,r') \leftarrow (4)$$

$$\nabla^2 \phi + k_0^2 \phi = f(r)$$

Volume Integral Equations: Solving

Get it into a form that we can solve: $\nabla^2(E - E_i) + k_0^2\epsilon_r E - k_0^2 E_i + k_0^2 E = k_0^2 E$

$$\checkmark\ \nabla^2\big(E(r) - E_i(r)\big) + k_0^2\big(E(r) - E_i(r)\big) = -k_0^2\big(\epsilon_r(r) - 1\big)E(r)$$

Using

$$\checkmark\ \nabla^2 g(r,r') + k_0^2 g(r,r') = -\delta(r,r')$$

we get

$$E(r) - E_i(r) = \int_{\underbrace{\ }_{V_1+V_2}} g(r,r')\, k_0^2\big(\epsilon_r(r') - 1\big)E(r')\,dr'$$

{Fredholm integral eqn of 2^{nd} kind.
Lipmann Schwinger eqn.}

known: $E_i(r)$, $\epsilon_r(r)$, $g(r,r')$, unknown $E(r)$

$$E(r) - \int_{\underbrace{\ }_{V_2}} g(r,r')\, k_0^2\big(\epsilon_r(r') - 1\big)E(r')\,dr' = E_i(r)$$

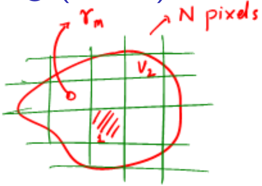
2 steps:
1) Find $E(r)$ inside V_2 $\rightarrow$ choose $r \in V_2$
2) Find $E(r)$ anywhere. $\rightarrow$ choose $r \in V_1$

$t_m(r) = \delta(r - r_m)$ (2D delta fn)

Volume Integral Equations: Solving (MoM)

Use MoM: Pulse basis, delta testing

$E(\bar{r}) = \sum_{n=1}^{N} \underbrace{a_n}_{\text{To solve for.}} \underbrace{p_n(\bar{r})}_{\text{2D pulses.}}$

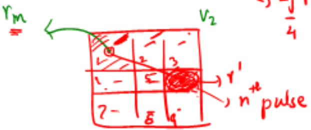


$\varepsilon_r(r) - 1 = \chi(r) = \sum_{n=1}^{N} \underbrace{\chi_n}_{\text{known.}} p_n(r)$

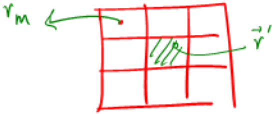
$E(r) - \int_{V_2} k_0^2\, g(r,r')\, \chi(r')\, E(r')\, dr' = E_i(r)$

1) Pulses: $\sum_{n=1}^{N} a_n p_n(r) - \int_{V_2} k_0^2\, g(r,r') \sum_{n=1}^{N} \chi_n a_n p_n(r')\, dr' = E_i(r)$

2) Testing: $\int (\ )\, \delta(r - r_m)\, dr \Rightarrow a_m - \int_{V_2} k_0^2\, \underbrace{g(r_m, r')}_{-\frac{j}{4} H_0^{(2)}(k_0|r' - r_m|)} \sum_{n=1}^{N} \chi_n a_n p_n(r')\, dr' = E_i(r_m)$



Volume Integral Equations: Solving (contd)



Any problems with singularities here?

2 cases

$r_m \notin n^{th}$ pulse $(m \neq n) \rightarrow$ quadrature. etc

$r_m \in n^{th}$ pulse $(m = n) \rightarrow$ potential singularity

$$\int_{n^{th}\text{ pulse}} g(r_m, \bar{r}')\, d\bar{r}'$$

$b^2 = \pi a^2$

$g \sim \ln x \quad x << 1$

$|r_m - r_n|$

$$\int_0^{2\pi}\int_0^{a} H_0^{(2)}(k\rho)\,\rho\, d\rho\, d\theta = \begin{cases} \frac{2\pi a}{k} J_1(ka)\, H_0^{(2)}(k\rho_{mn}) & m \neq n \\ \frac{2}{k^2}\left[\pi k a\, H_1^{(2)}(ka) - 2j\right] & m = n. \end{cases}$$

$|r_m - r'|$

$r' = r_n + \rho$

Volume Integral Equations: Summary

Putting it all together:

System:

step 1

$$a_m - \sum_{n=1}^{N} a_n \chi_n k_0^2 \int g(r_m, r')\, p_n(r')\, dr' = E_i(r_m)$$

$$Ax = b$$

$$x = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} \qquad b = \begin{bmatrix} E_i(r_1) \\ \vdots \\ E_i(r_N) \end{bmatrix}$$

step 2

$$E(r) = E_i(r) + k_0^2 \int_{V_2} g(r, r')\, \chi(r')\, E(r')\, dr'$$

$r \in V_1$

known

known.

Subs $E(r') = \sum_{n=1}^{N} a_n p_n(r')$

known.

total → incident + scattered.

Topics that were covered in this module

1. Motivation
2. Linear Vector Spaces
3. Formulating the Method of Moments
4. MoM: Surface Integral Equations
5. MoM: Volume Integral Equations

References:

Ch 2 of 'Integral Equation Methods for Electromagnetic and Elastic Waves', Chew, Tong, Hu

Ch 8 of 'Waves and fields in inhomogeneous media', Chew

Ch 2.5 of 'Computational Methods for Electromagnetics', Peterson, Ray, Mitra