Computational Electromagnetics : Introduction to Integral Equations

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Topics in this module

1. A simple line charge problem
2. Solving the Integral Equation

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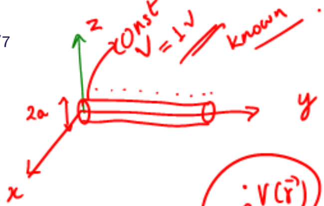

Electrostatic line charge

$V(\vec{r}')$

ρ

At any point:

line charge density

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathsf{L}} \frac{\boxed{\rho(\vec{r}')}}{R}\, dl'}$$

source pts

$|\vec{r} - \vec{r}'| =$ $R = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}$

obsv pt.

Steps: (1) Find ρ, (2) then V.

Where can we choose $\vec{r}$?

$\vec{r} \rightarrow$ to be along the y axis

$\vec{r} = (0, y, 0)$, $0 \le y \le L$

$$1 = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho(y')\,dy'}{R(y,y')}$$

Thus $R(y, y') = 0$ avoided.

Recap what is known/unknown. $\rightarrow$ ρ

Eig val $Ax = \lambda x$

Aside : Types of Integral Equations

Fixed limits of integration: Fredholm Integral Eqn

$f(x)$ may be zero or non-zero

homogeneous non-homogeneous

1st kind. only in

→ ❶ $f(x) = \int_a^b K(x,t)\,\psi(t)\,dt$

$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')\,dy'}{R(y,y')}$

2nd kind in & out.

→ ❷ $\psi(x) = f(x) + \lambda \int_a^b K(x,t)\,\psi(t)\,dt$

K : kernel, ψ : unknown

λ : unknown. f - known

Fredholm IE of first kind, non homogeneous

ψ : ρ, K : $\frac{1}{R(y,y')}$, f : V

One limit of integration is variable: Volterra Integral Eqn

❶ $f(x) = \int_a^x K(x,t)\,\psi(t)\,dt$

❷ $\psi(x) = f(x) + \lambda \int_a^x K(x,t)\,\psi(t)\,dt$

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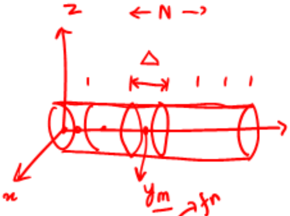
Forming a system of equations: discretization

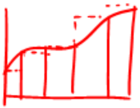


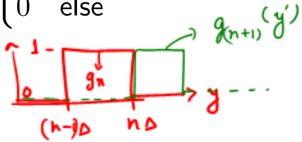
unknown scalars

"Basis fns" for ρ

Pulse fn

$$g_n(y') = \begin{cases} 1 & y' \in [(n-1)\Delta, n\Delta] \\ 0 & \text{else} \end{cases}$$





fn

Express $\rho(y') = \sum_{n=1}^{N} a_n g_n(y')$

$$\implies 4\pi\epsilon_0 = \sum_{n=1}^{N} a_n \int_0^l \frac{g_n(y')}{\sqrt{(y-y')^2+a^2}}dy'$$

Choose $y = y_m \to$ Matching point

$$4\pi\varepsilon_0 = a_1 \int_0^{\Delta} \frac{dy'}{R(y_m,y')} + a_2 \int_{\Delta}^{2\Delta} \frac{dy'}{R(y_m,y')} + \cdots + a_N \int_{(N-1)\Delta}^{l} \frac{dy'}{R(y_m,y')}$$

1 eqn, N variables $\rightarrow (a_1, a_2, \cdots, a_N) \rightarrow y_m$: general.

Forming a system of equations: discretization

Continue the process for *all* matching points:

$$\to A_{mn} = \int_{(n-1)\Delta}^{n\Delta} \frac{g_n(y')}{R(y_m, y')} dy'$$

matching pt (row)

source pt (col no)

$b_m = 4\pi\epsilon_0$

$$4\pi\epsilon_0 = a_1 \int_0^{\Delta} \frac{dy'}{R(y_1, y')} + \cdots + a_N \int_{(N-1)\Delta}^{\ell} \frac{dy'}{R(y_1, y')}$$

$$\vdots$$

$$4\pi\epsilon_0 = a_1 \int_0^{\Delta} \frac{dy'}{R(y_N, y')} + \cdots + a_N \int_{(N-1)\Delta}^{\ell} \frac{dy'}{R(y_N, y')}$$

N eqns, N vars.

1) Direct method → Gaussian Elim

A \ b

2) Indirect

N?

Can be combined into:

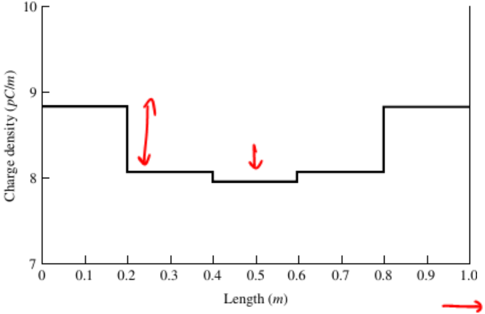
$$\begin{pmatrix} b \end{pmatrix}_{N\times 1} = \underset{N\times N}{A} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}_{N\times 1} \Rightarrow Ax = b$$

x → unknown.

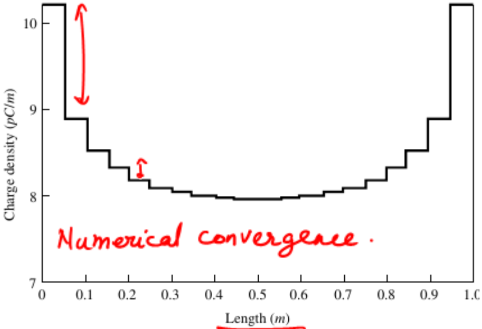
Numerical aspects: how to choose N?

Credit: Balanis, Antenna Theory & Design

ρ



(a) $N = 5$



(b) $N = 20$

$\longrightarrow y$

$N \to M > N$

- Sufficiently large N : Relative change criteria

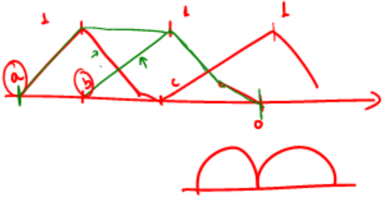
$$\frac{\| x^{(N)} - x^{(M)} \|_2}{\| x^{(N)} \|_2} < \varepsilon$$

$\varepsilon = 0.1, \; 0.01$

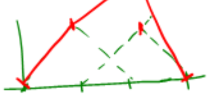
Soln obt when N segs chosen

Improving accuracy by changing basis functions

Sub-domain basis functions:



$1\,f_a(y) + 2\,f_b(y)$



piecewise linear approx.

Full-domain basis functions:

$$f(y) = a_0 + \sum_{n=1}^{N} a_n \cos(kny) + b_n \sin(kny)$$

Topics that were covered in this module

1. A simple line charge problem

2. Solving the Integral Equation

Reference: Ch 8.2 of Antenna Theory & Design by Balanis