Computational Electromagnetics : Antenna computations

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Topics in this module

$$\begin{tikzcd}
\vec{J}, \vec{M} \arrow[rr, "1"] \arrow[dr, "2"'] & & \vec{E}, \vec{H} \\
& \underline{\vec{A}, \vec{F}} \arrow[ur, "2"'] &
\end{tikzcd}$$

1. Scalar and Vector Potentials
2. The Simplest Antenna
3. Finite Antennas & Integral Equations

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$e^{j\omega t}$

Electromagnetics problems: scalar and vector potentials

$\nabla\times\vec{E} = -j\omega\mu\bar{H}$ (1), $\nabla\times\vec{H} = \vec{J} + j\omega\varepsilon\bar{E}$ (2), $\nabla\cdot\bar{B} = 0$ (3), $\nabla\cdot\bar{D} = \rho$ (4)

$\bar{B} = \mu\bar{H}$

$\nabla\cdot\bar{B} = 0$ (1) Div of curl $= 0$

$\Rightarrow \nabla\cdot\bar{B} \Rightarrow \vec{H} = \frac{1}{\mu}\nabla\times\bar{A}$ — (5)

#(2) Any vector field is completely specified (upto a const) by its curl & div.

Combine (1) & (5)

$\nabla\times\vec{E} = -j\omega\nabla\times\vec{A}$

$\nabla\times[\vec{E} + j\omega\vec{A}] = 0$ — (6)

If $\vec{E} + j\omega\vec{A}$ is $-\nabla\phi$ then (6) always true.

$\Rightarrow \vec{E} = -j\omega\vec{A} - \nabla\phi$ — (7)

$\vec{A}$: magnetic vector potential

ϕ : scalar potential.

The Lorentz gauge and the vector wave equation

① $\nabla\times\bar{E} = -j\omega\mu\bar{H} \xrightarrow{\text{curl}} \nabla\times(\nabla\times\bar{E}) = -j\omega\,\nabla\times\mu\bar{H} = -j\omega\,\nabla\times(\nabla\times A)$

② $\nabla\times\bar{H} = J + j\omega\varepsilon E \longrightarrow \frac{1}{\mu}\nabla\times(\nabla\times A) = \bar{J} + j\omega\varepsilon\left[-j\omega\bar{A} - \nabla\phi\right]$

$$\nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A} = \mu\bar{J} + \omega^2\mu\varepsilon\bar{A} - j\omega\varepsilon\mu\nabla\phi$$

like a vector wave eqn

$$\nabla^2\bar{A} + \omega^2\mu\varepsilon\bar{A} - \nabla\left[\nabla\cdot\bar{A} + j\omega\varepsilon\mu\phi\right] = -\mu\bar{J} \quad -③$$

Now specify the div of $\vec{A}$: $\nabla\cdot\bar{A} = -j\omega\varepsilon\mu\Phi$] Lorentz gauge. → choice of div.

[Aharanov Bohm effect]

$$\nabla^2\vec{A} + \underbrace{\omega^2\mu\varepsilon}\vec{A} = -\mu\bar{J} \quad -④$$

$\nabla^2 G + k^2 G = -\delta$ → Green's fn.

$$\vec{A}(r) = \iiint \mu J(r')\,G(r,r')\,d\bar{r}' \quad ⑤$$

By integral Eqn theory

$$\Phi = \frac{j}{\omega\mu\varepsilon}\nabla\cdot\bar{A}$$

Flow of problem solving in antenna problems

1) Given $\vec{J}$ $\rightarrow$ $\vec{A} = \mu \vec{J} \otimes G$

2) Obtain $\vec{H}$ $\Rightarrow$ $\vec{H} = \frac{1}{\mu} \nabla \times \vec{A}$

3) Obtain $\vec{E}$ $\Rightarrow$ $\vec{E} = -j\omega\vec{A} - \nabla\phi = -j\omega\vec{A} - \nabla\left(j\,\frac{\nabla\cdot\vec{A}}{\omega\mu\epsilon}\right)$

$$\nabla\times\vec{H} = \vec{J} + j\omega\epsilon\vec{E}$$

Same $\bar{E}, \bar{H}$ regardless of $\vec{A}, \phi$

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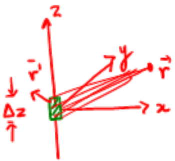
The Hertz Dipole: $\vec{A}$



$$\vec{J}(x,y,z) = \begin{cases} I\,\delta(x)\delta(y)\,\hat{z} & \text{for } -\frac{\Delta z}{2} < z < \Delta z/2 \\ 0 & \text{else} \end{cases}$$

$$\vec{G}_{3D}(r,r') = \frac{e^{-jk\underbrace{|r-r'|}_{R}}}{4\pi|r-r'|}$$

$\underline{\Delta z << 1}$

① $\Rightarrow \vec{A} = \mu \vec{J} \circledast G_{3D}$

$$\vec{A}(r) = \hat{z}\mu \int_{-\Delta z/2}^{\Delta z/2} \frac{I e^{-jkR}}{4\pi R} dr' \qquad R = |\vec{r}-\vec{r}'|$$

$$\vec{A} = \hat{z}\, \frac{\mu I \Delta z\, e^{-jkr}}{4\pi r}$$

where $r = |\vec{r}|$

② For $\vec{H} = \frac{1}{\mu}\nabla\times\vec{A} = \frac{1}{\mu}\nabla\times(A_z\hat{z})$

$$= \frac{1}{\mu}\Big[\nabla A_z \times \hat{z} + A_z \underbrace{\nabla\times\hat{z}}_{=0}\Big]$$

V.C Identity

$=0$ curl of a const

$$= \frac{1}{\mu}\left(\nabla A_z \times \hat{z}\right)$$

The Hertz Dipole: $\vec{H}, \vec{E}$

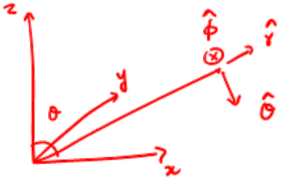
$\nabla \rightarrow \frac{\partial}{\partial r}\hat{r}$

$fn(r)$

$$\vec{H} = \frac{1}{\mu}\nabla A_z \times \hat{z}$$

$$= \frac{I\Delta z}{4\pi}\left[\frac{\partial}{\partial r}\left(\frac{e^{-jkr}}{r}\right)\hat{r} \times \hat{z}\right]$$

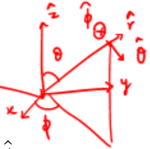
$$= \frac{I\Delta z}{4\pi}\left[\frac{-jk}{r} - \frac{1}{r^2}\right]e^{-jkr}\underbrace{\hat{r}\times\hat{z}}_{-\sin\theta\,\hat{\phi}}$$



far away fields $\propto \frac{1}{r}$.

$$\vec{H} = \frac{I\Delta z}{4\pi}\left[\frac{jk}{r} + \frac{1}{r^2}\right]e^{-jkr}\sin\theta\,\hat{\phi}$$

$$\vec{E} = \frac{1}{j\omega\varepsilon}\nabla\times\vec{H}$$ (away from source)



The Hertz Dipole: Far fields

$$\vec{H} = \frac{I\Delta z}{4\pi} jk(1 + \frac{1}{jkr})\frac{e^{-jkr}}{r}\sin\theta\hat{\phi}$$

$$\vec{E} = \frac{I\Delta z}{4\pi} j\omega\mu[1 + \frac{1}{jkr} + \frac{1}{(jkr)^2}]\frac{e^{-jkr}}{r}\sin\theta\hat{\theta}$$

$$+\frac{I\Delta z}{2\pi} j\omega\mu[\frac{1}{jkr} + \frac{1}{(jkr)^2}]\frac{e^{-jkr}}{r}\cos\theta\hat{r}$$

$kr >> 1.$

$$\vec{H} = \frac{I\Delta z}{4\pi} jk \frac{e^{-jkr}}{r} \sin\theta\, \hat{\phi}$$

$$\vec{E} = \frac{I\Delta z}{4\pi} j\omega\mu \frac{e^{-jkr}}{r} \sin\theta\, \hat{\theta}$$

TEM wave

radiation fields

Purely real.

radiated power.

Far field, fields $\propto \frac{1}{r}$

$$\vec{S} = \frac{1}{2} \vec{E} \times \vec{H}^* = \frac{1}{2}\left(\frac{I\Delta z}{4\pi}\right)^2 \frac{k\omega\mu \sin^2\theta}{r^2}\hat{r}$$

$$P_T = \iint \vec{S}\cdot\underbrace{\vec{ds}}_{r^2\sin\theta\, d\theta\, d\phi} = \text{indepn of } r$$

$$\frac{|E|}{|H|} = \eta$$ Characteristic impedance.

The Hertz Dipole: Near fields

$$\vec{H} = \frac{I\Delta z}{4\pi} jk\left(1+\frac{1}{jkr}\right)\frac{e^{-jkr}}{r}\sin\theta\hat{\phi}$$

$$\vec{E} = \frac{I\Delta z}{4\pi} j\omega\mu\left[1+\frac{1}{jkr}+\frac{1}{(jkr)^2}\right]\frac{e^{-jkr}}{r}\sin\theta\hat{\theta}$$

$$+\frac{I\Delta z}{2\pi} j\omega\mu\left[\frac{1}{jkr}+\frac{1}{(jkr)^2}\right]\frac{e^{-jkr}}{r}\cos\theta\hat{r}$$

$kr << 1$

$$\rightarrow \vec{H} = \frac{I\Delta z}{4\pi}\frac{e^{-jkr}}{r^2}\sin\theta\,\hat{\phi} \rightarrow \frac{1}{r^2}$$

$$\vec{E} = \frac{I\Delta z\, j\omega\mu}{4\pi}\left[\frac{1}{(jkr)^2}\sin\theta\,\hat{\theta}\right]\frac{e^{-jkr}}{r}$$

$$+\frac{I\Delta z}{2\pi} j\omega\mu\left[\frac{1}{(jkr)^2}\cos\theta\,\hat{r}\right]\frac{e^{-jkr}}{r} \rightarrow \frac{1}{r^3}$$

$$\vec{S}^{nf} = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{r^5}\left(\sin^2\theta\,\hat{r} - \sin 2\theta\,\hat{\theta}\right)(j(\,))$$

$\rightarrow \frac{1}{r^5}$, purely imag.

Energy transferred between E & H fields $\longleftarrow$ $\rightarrow$ Reactive Fields.

The Hertz Dipole: Visualizing fields

$\vec{H} \propto (\ )\ \hat{\phi}$

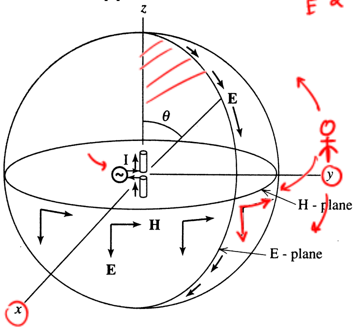
$\vec{E} \propto (\ )\ \hat{\theta}$

(r,θ)

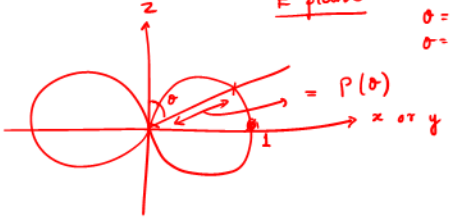
$\theta = 0 \quad (0,0)$

$\theta = \pi/2 \quad (1, \pi/2)$

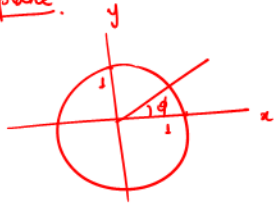
Cr: Stutzman [1]



E plane



H plane.



Field pattern: $\underline{F}(\theta, \phi) = \dfrac{E_\theta}{E_\theta(\text{max})} = \sin\theta$

at a given r

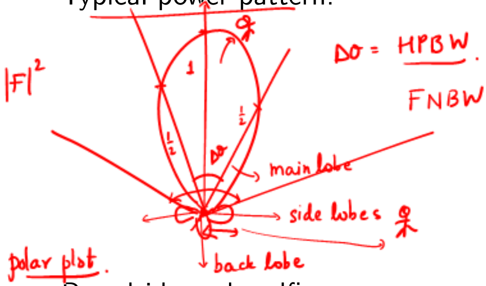
i.e. on a given sphere.

Power.

$$\underline{P = |F(\theta,\phi)|^2}$$

Antenna patterns in general

Typical power pattern:



Side lobe level (SLL)

$$\text{SLL}_{dB} = 20\log\frac{F(\text{SLL})}{F(\text{max})}$$

Minimize for best perfomance.

Broadside and endfire

NF v/s FF: $\frac{2D^2}{\lambda}$

D: Max dimension of the ant.

λ: wavelength.

$\frac{2\times 1}{0\cdot 3} \sim 6-7\text{m}$

Antenna modelling: the scattered field

[Radius $= a \ll \lambda$ → Assumption.
z-directed currents →

$\vec{A}' = \mu \vec{J} * G = A_z \hat{z}$

Lorentz gauge condn.

Recall: $\phi = \frac{j}{\omega\epsilon_0\mu_0}\nabla\cdot\vec{A}$ and $\vec{E} = -j\omega\vec{A} - \nabla\phi$ ←

$\vec{H}' = \frac{1}{\mu}\nabla\times\vec{A}'$

$$\phi = \frac{j}{\omega\mu_0\epsilon_0}\frac{\partial A_z}{\partial z}$$

$$E_z = -j\omega A_z - \frac{j}{\omega\mu_0\epsilon_0}\frac{\partial^2 A_z}{\partial z^2} = \frac{1}{j\omega\mu_0\epsilon_0}\left(k^2 A_z + \frac{\partial^2 A_z}{\partial z^2}\right)$$

$$A_z = \mu J_z * G \rightarrow \frac{e^{-jkR}}{4\pi R} \Rightarrow E_z^s = \frac{1}{j\omega\mu_0\epsilon_0}\iiint\left(\frac{\partial^2}{\partial z^2}G + k^2 G\right) J\, dv' \quad — (1)$$

$E_z^s(r)$ — due to induced current.

$G(r,r')$ ✓

$J(r')$ ×

?✓ $= -E_z^{in}$

Inside & on surface

$E_{Tot} = 0 = E_{inc} + E_z^s$

Fredholm intg eqn 1st kind ← (1)

induced current produces a scattered field

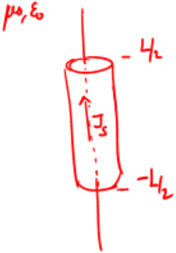
Antenna modelling: Pocklington's equation

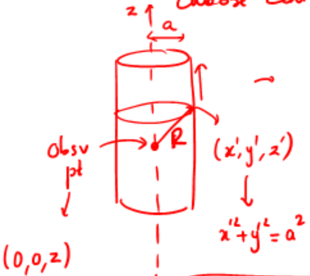
[Assume perfect electric conductor
⇒ surface currents]

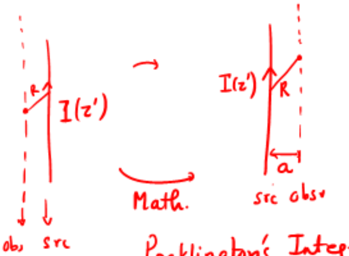
$$E_z^s = -E_z^{in} = \frac{1}{j\omega\varepsilon_0} \oint \int_{-L/2}^{L/2} \left(\frac{\partial^2 G}{\partial z^2} + k^2 G\right) J_s \, \underline{dz'\, d\phi'} \quad — ②$$

$$\oint J_s \, d\phi' = I(z')$$

$r \in$ conductor
choose centre pt (axis). $R = \sqrt{a^2 + (z-z')^2}$







Pocklington's Integral Eqn.

$$\boxed{-E_z^{in} = \frac{1}{j\omega\varepsilon_0} \int_{-L/2}^{L/2} \underbrace{\left(\frac{d^2 G(z,z')}{dz^2} + k^2 G(z,z')\right)}_{K(z,z')} \underline{\underline{I(z')}} \, \underline{dz'}}$$

Pocklington's equation: Solution using MoM

$$\left[-\int_{-L/2}^{L/2} I(z')\,K(z,z')\,dz' = E_z^i(z) \right]$$

Step 1 → $I(z') = \sum_{n=1}^{N} I_n \underbrace{F_n(z')}_{\text{pulse}}$ pulse basis

Step 2 → testing by delta fn $\int dz\,\delta(z-z_m)\,[\text{Eqn}]$ delta testing

$$-\int_{-L/2}^{L/2} \sum_{n=1}^{N} I_n F_n(z')\,K(z_m,z')\,dz' = E_z^i(z_m)$$

repeat for z_m, $m = 1, \dots, N$

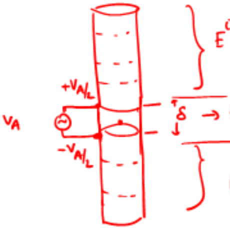
Step 3, solve for I_n's.

basis – triangular, fourier series

Source Modelling

T_x R_x

Delta-gap

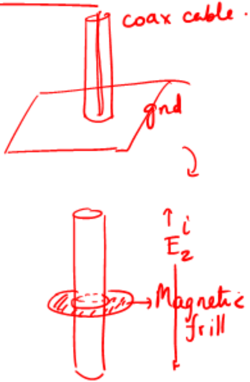


$E^i = 0$

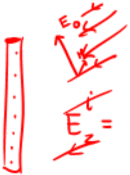
$E^i = \frac{V_A}{\delta}$

$E^i = 0$

Magnetic Frill

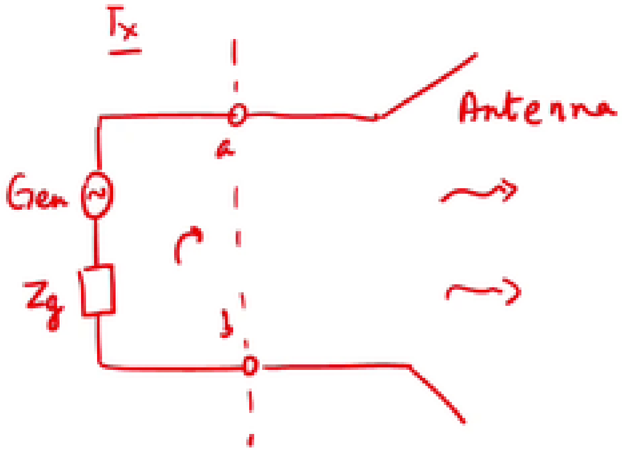


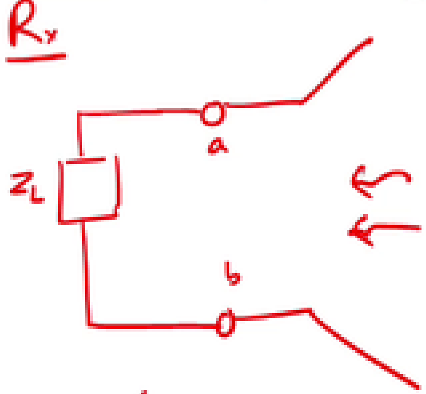
Incident wave

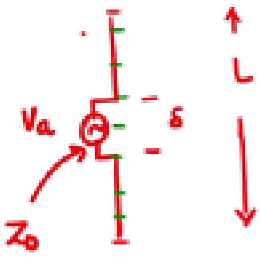


$E_z^i = \hat{z} \cdot \vec{E}_0 e^{-jk_0 z\cos\theta}$

Circuit model of antenna, computing input impedance

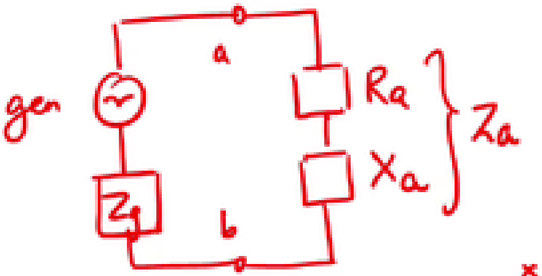


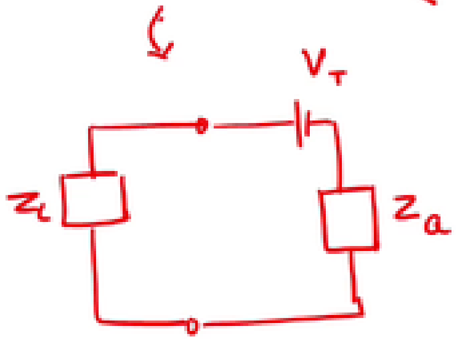




After MoM $\Rightarrow$ I is known.

$$Z_a = V_a / I(z_0)$$





optimal power tx: $Z_g = Z_a^*$

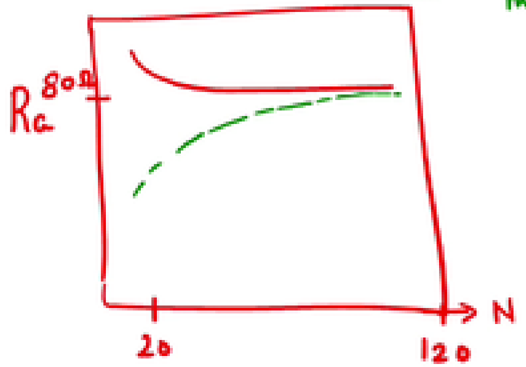
What is $\boxed{Z_a}$?

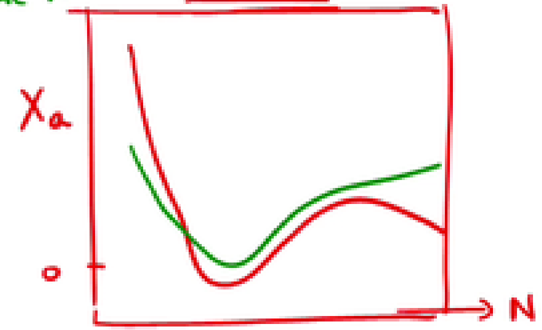
$Z_a = R_a + jX_a \rightarrow \underline{L = 0.47\lambda}, \quad a = 0.005\lambda$

MoM: Details matter

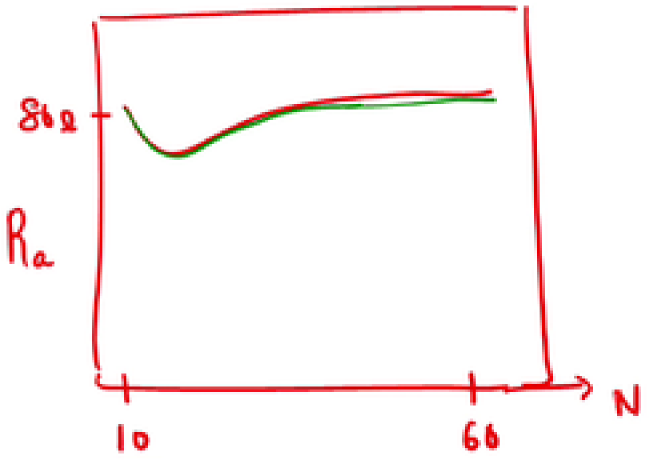
— delta gap
— magnetic frill

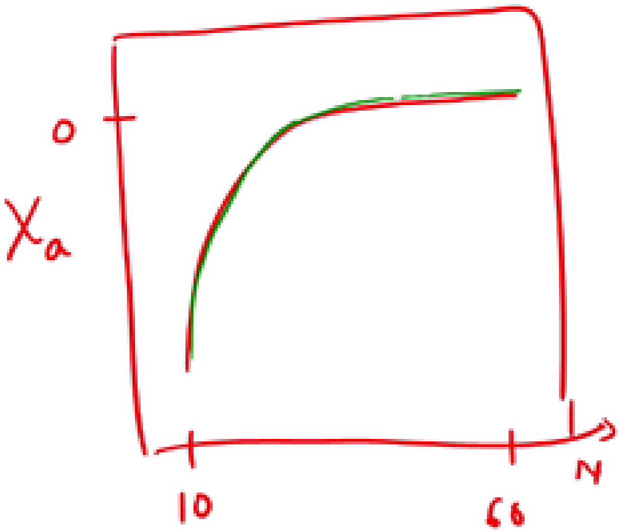
Pulse-basis,
delta-testing





Pulse-basis,
pulse-testing

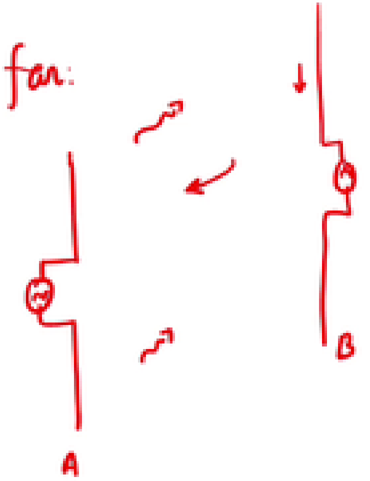




Two antennas: A circuit model

So far:

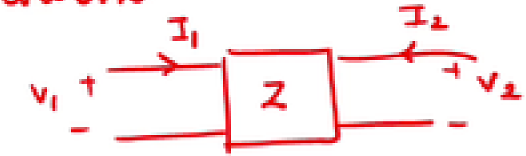
active or passive



Single ant: $E_z^i(r) + E_z^s(r) = 0$ for $r \in A$

Both ant: $E_z^i(r) + E_z^{s,A}(r) + E_z^{s,B}(r) = 0$ for $r \in A$

2 port network.



$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

$Z_{11}, Z_{22} \rightarrow$ self impedances

$Z_{21}, Z_{12} \rightarrow$ Mutual impedances.

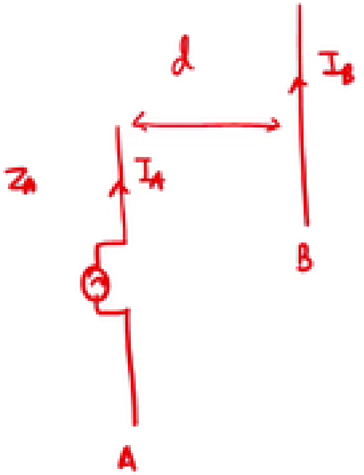
$$\frac{V_1}{I_1} = Z_{1d} = Z_{11} + Z_{12}\frac{I_2}{I_1}$$
$$\frac{V_2}{I_2} = Z_{2d} = Z_{21}\left(\frac{I_1}{I_2}\right) + Z_{22}$$

$\rightarrow$ Driving point impedance

CEM $\rightarrow$ Calc. exactly.

Assumptions: both z-directed, radius << λ

Two antennas: Boundary conditions



Single antenna:

$$E_z^{i,A}(r) + E_z^{S,A}(r) = 0 \quad \text{for } r \in A$$

Two antennas:

$$1) \quad \underbrace{E_z^{i,A}(r)} + \underline{E_z^{S,A}(r)} + \underline{E_z^{S,B}(r)} = 0, \quad \text{for } r \in A$$

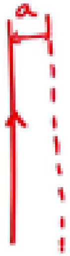
$$2) \quad E_z^{S,A}(r) + E_z^{S,B}(r) = 0, \quad \text{for } r \in B$$

(assume B has no source, else add $E_z^{i,B}(r)$ to LHS)

Null boundary conds.

$$\int_{-L_A/2}^{L_A/2} I_A(z') K(z,z') dz' = -E_z^i(z)$$

To avoid



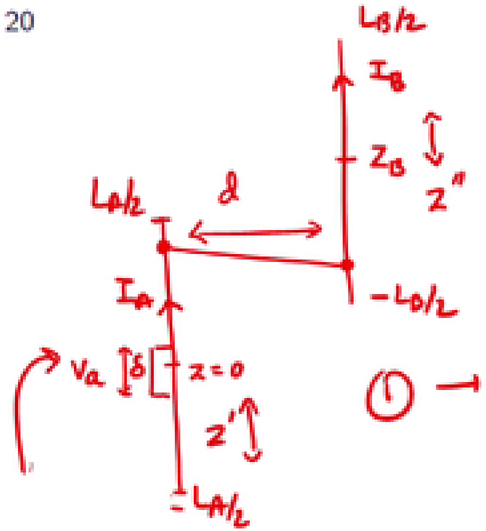
How did we solve single ant? P.I.E

$$\frac{1}{j\omega\varepsilon_0} \int_{-L_A/2}^{L_A/2} I_A(z') \overbrace{\left(\frac{\partial^2}{\partial z^2} + k^2\right) G(z,z')}^{j\omega\varepsilon_0 K(z,z')} dz' = -E_z^i(z)$$

$$G(z,z') = \frac{e^{-jkR}}{4\pi R}$$

$$\Rightarrow \quad R = \sqrt{\underline{\underline{a}}^2 + (z-z')^2}$$

Two antennas: Formulating the equations



B.C: $E_z^{S,A}(z) + E_z^{S,B}(z) = -E_i^{z,A}(z)$ — Seltagap

① → $\displaystyle\int_{-L_A/2}^{L_A/2} I_A(z')\underset{A\to A}{K}(z,z')\,dz' + \int_{-L_B/2}^{L_B/2} I_B(z'')\underset{B\to A}{K}(z,z'')\,dz'' = -\underbrace{E_i^{z,A}(z)}$, $z \in A$

$\boxed{K_{src \to obsv}}$

② → $\displaystyle\int_{-L_A/2}^{L_A/2} I_A(z')\underset{A\to B}{K}(z,z')\,dz' + \int_{-L_B/2}^{L_B/2} I_B(z'')\underset{B\to B}{K}(z,z'')\,dz'' = 0$, $z \in B$.

I_A, I_B are unknown.

$Z_{A,d} = \dfrac{V_A}{I_A(z=0)}$ Driving pt.

In $K_{A\to A}$ or $K_{B\to B}$:

$R = \sqrt{(z-z')^2}$ or $\sqrt{(z-\underbrace{y''}_{z_B+z''})^2}$.

In $K_{A\to B}$ or $K_{B\to A}$

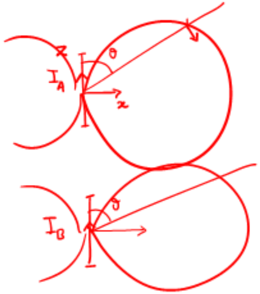
$R = \sqrt{d^2 + (z - \underbrace{(z_B+z'')}))^2}$

$$\begin{bmatrix} \overbrace{F_{AA}}^{N_A\times N_A} & F_{BA} \\ F_{AB} & \underbrace{F_{BB}}_{N_B\times N_B} \end{bmatrix} \begin{bmatrix} I_A \\ I_B \end{bmatrix} = \begin{bmatrix} h \\ 0 \end{bmatrix}$$

$(N_A+N_B)\times(N_A+N_B)$

Hertz dipole: $\vec{E}_0(\theta) = \underbrace{\frac{c_0 I e^{-jkr}}{r} \sin\theta \, \hat{\theta}}_{}$. Assume: User in far field.

Two antennas: any gain? (pun intended)



Net $\vec{E}$ field?

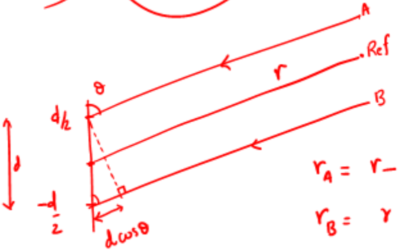
$$\vec{E} = \underbrace{\vec{E}_1}_{} + \underbrace{\vec{E}_2}_{} \quad \text{(superposition)}$$

Also assume no mutual coupling

$$\vec{E} = \frac{c_0}{r} I \left(e^{-jkr_A} + e^{-jkr_B} \right) \sin\theta \, \hat{\theta}$$

r: Num & Den

$$\vec{E} = \underbrace{\frac{c_0 I e^{-jkr}}{r} \left(2\cos\left(\frac{kd\cos\theta}{2} \right) \right) \sin\theta \, \hat{\theta}}_{}$$



$r_A = r - \frac{d}{2}\cos\theta$

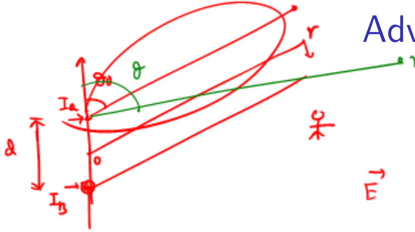
$r_B = r + \frac{d}{2}\cos\theta$

Check at $\theta = 0^\circ$, $\vec{E} = 0$

$\theta = \pi/2$, $\vec{E} = 2\vec{E}_0(\pi/2)$ ✓

Phased array antennas.

Advantage of two antennas: beam forming/steering.



$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{C_0}{r}\left[I_a e^{-jkr_A} + I_B e^{-jkr_B} \right] \sin\theta\, \hat{\theta}$$

$$\vec{E} = \frac{C_0}{r} e^{-jkr}\left[I_a e^{\underbrace{+jk\frac{d\cos\theta}{2}}} + I_b e^{-jk\frac{d\cos\theta}{2}} \right] \sin\theta\, \hat{\theta}$$

$$I_a = I_0 e^{-jk\frac{d\cos\theta_0}{2}}, \quad I_b = I_0 e^{+jk\frac{d\cos\theta_0}{2}}$$

$$\Rightarrow \vec{E} = \frac{C_0 e^{-jkr}}{r}\left[I_0 \cdot \times 2 \right] \sin\theta\, \hat{\theta}$$

extra phase is added by a phase shifter.

↳ Say, beam max at $\theta = \theta_0$ $(\theta_0 \neq \pi/2)$, what is $\vec{E}(\pi/2)$?

$$\vec{E} = \frac{C_0}{r} e^{-jkr}\left[I_0 e^{-jk\frac{d\cos\theta_0}{2}} \underbrace{e^{jk\frac{d\cos\theta}{2}}}_{1} + I_0 e^{jk\frac{d\cos\theta_0}{2}} \underbrace{e^{-jk\frac{d\cos\theta}{2}}}_{1} \right] ( )$$

$$= \frac{C_0}{r} e^{-jkr} I_0 \underbrace{\left[2\cos\left(\frac{kd\cos\theta_0}{2}\right) \right]} \underbrace{\sin\theta}_{1}\, \hat{\theta} \quad (\text{at } \theta = \pi/2)$$

Topics that were covered in this module

1. Scalar and Vector Potentials
2. The Simplest Antenna
3. Finite Antennas & Integral Equations
4. Mutual coupling between two antennas & antenna arrays.

References:

(1) Ch 1,10 of Antenna Theory & Design, Stutzman and Thiele, Wiley ✓

(2) Ch 3,8 of Antenna Theory & Design, C A Balanis, Wiley

(3) Mutual coupling between a wire antenna of finite conductivity and a large object, Vossen, Masters Thesis, Eindhoven University of Technology, 1997 https://pure.tue.nl/ws/portalfiles/portal/46983690/684720-1.pdf