Applications of Computational Electromagnetics: Microwave Inverse Imaging

Uday K Khankhoje

Electrical Engineering, IIT Madras

Table of Contents

Inverse Imaging : What is it?

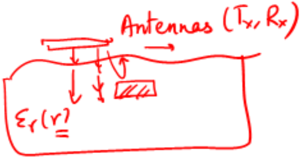
Inverse Problems:

This is different.
Given scattered fields, $\vec{E}_s(\vec{r})$, tell me what is $\epsilon_r(\vec{r})$?
Problem has no unique solution.

E.g. buried land mine detection
structural health monitoring
breast cancer detection, etc.

Forward Problems:

We are used to these.
Given permittivity, $\epsilon_r(\vec{r})$, find the radiated or scattered fields in a problem.
Problem has a unique solution.



Breast Cancer in India: a crisis

Context

A 2017 study conducted by the National Institute of Pathology in India[1]

- Ranked breast cancer as having the highest rate of incidence and mortality among Indian women (earlier occupied by cervical cancer)
- Mortality to incidence ratio: as high as 66 in rural areas, around 8 in urban settings

[1] S. Malvia et al, "Epidemiology of breast cancer in Indian women," Asia-Pacific Journal of Clinical Oncology, 2017.

Breast Cancer in India: a crisis

Context

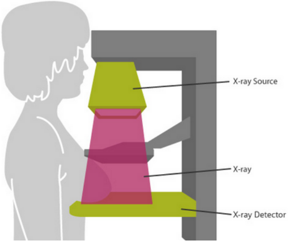
A 2017 study conducted by the National Institute of Pathology in India[1]

- Ranked breast cancer as having the highest rate of incidence and mortality among Indian women (earlier occupied by cervical cancer)
- Mortality to incidence ratio: as high as 66 in rural areas, around 8 in urban settings
- Lack of diagnostic aids has been identified as being responsible for these statistics
- Sharp divide between rural and urban survival rates – issues in accessibility and affordability of diagnostic devices.

[1] S. Malvia et al, "Epidemiology of breast cancer in Indian women," Asia-Pacific Journal of Clinical Oncology, 2017.

Can Microwave Technology Help?

Current methods are expensive, time consuming, inaccessible (MRI screening), or cause cancer (X-ray)

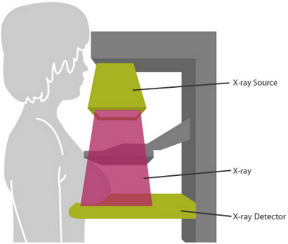


Can Microwave Technology Help?

Breast MRI

Photo Courtesy of GE Healthcare

Current methods are expensive, time consuming, inaccessible (MRI screening), or cause cancer (X-ray)



Need methods that are: safe, inexpensive, quick, and non invasive
Microwave (RF) technology has the potential!

- RF waves penetrate human tissues without causing ionizing damage ✓
- RF components (in the 1-10GHz range) are cheap due to other popular applications such as telecom, WiFi, etc

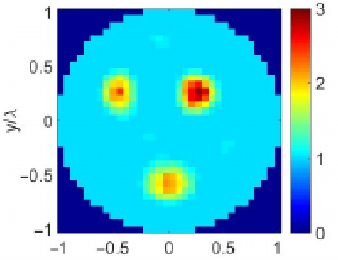
Underlying Principle: waves are scattered by obstacles

High school experiment on prisms: light gets reflected & transmitted (bent) on hitting an object (glass) of different refractive index

Underlying Principle: waves are scattered by obstacles

High school experiment on prisms: light gets reflected & transmitted (bent) on hitting an object (glass) of different refractive index

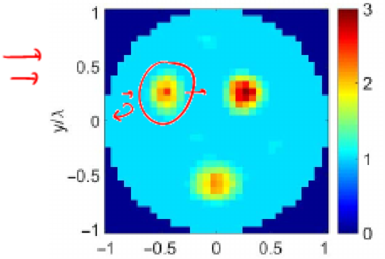
When microwave travels through tissue → gets scattered by different constituents (blood, fat, cancer).



Underlying Principle: waves are scattered by obstacles

High school experiment on prisms: light gets reflected & transmitted (bent) on hitting an object (glass) of different refractive index

When microwave travels through tissue → gets scattered by different constituents (blood, fat, cancer).



Cancerous tissue has different permittivity than healthy
→ scatters microwaves differently
→ fields encode information of scattering objects

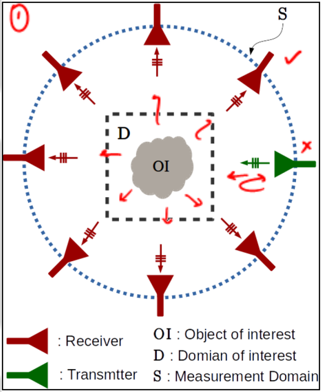
Breast Cancer Detection: High Level Idea

Data collection

- Surround object by Tx/Rx
- One Tx ON, all others Rx; store fields

Processing

- Use fields to solve mathematical problem to get permittivity as a function of space, $\epsilon(r)$
- Look up tables of ϵ values prepared by biologists to infer cancer



Diagnosis

Look up tables of ϵ values prepared by biologists to infer cancer

Table of Contents

Maxwell's equations that we know and love[2]! Vol. Intg.

$$\nabla \times \vec{E}(r) = -j\omega\mu\,\vec{H}(r), \quad \nabla \times \vec{H}(r) = j\omega\,\epsilon(r)\,\vec{E}(r) + \vec{J}(r) \tag{1}$$

[2] Single frequency ($e^{j\omega t}$), two-dimensions ($x - y$), single polarization (E_z) TM

Maxwell's equations that we know and love[2]!

$$\nabla \times \vec{E}(r) = -j\omega\mu\, \vec{H}(r), \quad \nabla \times \vec{H}(r) = j\omega\, \epsilon(r)\, \vec{E}(r) + \vec{J}(r) \tag{1}$$

Combine these equations using vector calculus into a wave equation

$$\nabla^2 E_z(r) + k_0^2 \epsilon_r(r) E_z(r) = j\omega\mu J_z(r) \tag{2}$$

[2] Single frequency ($e^{j\omega t}$), two-dimensions ($x - y$), single polarization (E_z)

Maxwell's equations that we know and love[2]!

$$\nabla \times \vec{E}(r) = -j\omega\mu\,\vec{H}(r), \quad \nabla \times \vec{H}(r) = j\omega\,\epsilon(r)\,\vec{E}(r) + \vec{J}(r) \tag{1}$$

Combine these equations using vector calculus into a wave equation

$$\nabla^2 E_z(r) + k_0^2 \epsilon_r(r) E_z(r) = j\omega\mu J_z(r) \tag{2}$$

Specialize this to two cases [without and with object $\epsilon_r(r)$]:

$$\nabla^2 E_i(r) + k_0^2 E_i(r) = j\omega\mu J(r) \quad E_i : \text{incident field} \tag{3}$$

$$\nabla^2 E(r) + k_0^2 \epsilon_r(r) E(r) = j\omega\mu J(r) \quad E : \text{total field} \tag{4}$$

[2] Single frequency ($e^{j\omega t}$), two-dimensions ($x - y$), single polarization (E_z)

Maxwell's equations that we know and love[2]!

$$\nabla \times \vec{E}(r) = -j\omega\mu\, \vec{H}(r), \quad \nabla \times \vec{H}(r) = j\omega\, \epsilon(r)\, \vec{E}(r) + \vec{J}(r) \tag{1}$$

Combine these equations using vector calculus into a wave equation

$$\nabla^2 E_z(r) + k_0^2 \epsilon_r(r) E_z(r) = j\omega\mu J_z(r) \tag{2}$$

Specialize this to two cases [without and with object $\epsilon_r(r)$]:

$$\nabla^2 E_i(r) + k_0^2 E_i(r) = j\omega\mu J(r) \quad E_i : \text{incident field} \tag{3}$$

$$\nabla^2 E(r) + k_0^2 \epsilon_r(r) E(r) = j\omega\mu J(r) \quad E : \text{total field} \tag{4}$$

Subtract the two (eliminate source currents) + some algebra

$$\nabla^2 [E(r) - E_i(r)] + k_0^2 [E(r) - E_i(r)] = -k_0^2 (\epsilon_r(r) - 1) E(r) \tag{5}$$

Define contrast $\chi(r) = (\epsilon_r(r) - 1)$

[2]Single frequency ($e^{j\omega t}$), two-dimensions ($x - y$), single polarization (E_z)

Processing the Wave Equation into an Integral equation

$$\nabla^2[E(r) - E_i(r)] + k_0^2[E(r) - E_i(r)] = -k_0^2\,\chi(r)\,E(r)$$

Processing the Wave Equation into an Integral equation

$$\nabla^2[E(r) - E_i(r)] + k_0^2[E(r) - E_i(r)] = -k_0^2\,\chi(r)\,E(r)$$

Forward problem Given $E_i(r)$, $\epsilon_r(r)$ obtain $E(r)$ everywhere
Unique solution, all commercial CEM codes

Inverse problem Given $E(r)$, $E_i(r)$ obtain $\epsilon_r(r)$ everywhere
Infinite solutions, need apriori info!

Processing the Wave Equation into an Integral equation

$$\nabla^2[E(r) - E_i(r)] + k_0^2[E(r) - E_i(r)] = -k_0^2\,\chi(r)\,E(r)$$

Forward problem Given $E_i(r)$, $\epsilon_r(r)$ obtain $E(r)$ everywhere
Unique solution, all commercial CEM codes

Inverse problem Given $E(r)$, $E_i(r)$ obtain $\epsilon_r(r)$ everywhere
Infinite solutions, need apriori info!

We know how to solve this!

- Use theory of integral equations and Green's functions
- *Suppose* you knew the solution to this problem:
  $\nabla^2 G(r, r') + k^2 G(r, r') = -\delta(r, r')$ [impulse resp]
  δ is a Dirac delta function

Processing the Wave Equation into an Integral equation

$$\nabla^2[E(r) - E_i(r)] + k_0^2[E(r) - E_i(r)] = -k_0^2\,\chi(r)\,E(r)$$ ①

Forward problem Given $E_i(r)$, $\epsilon_r(r)$ obtain $E(r)$ everywhere
  Unique solution, all commercial CEM codes
Inverse problem Given $E(r)$, $E_i(r)$ obtain $\epsilon_r(r)$ everywhere
  Infinite solutions, need apriori info!

$\nabla^2 + k_0^2$

We know how to solve this!

- Use theory of integral equations and Green's functions
- *Suppose* you knew the solution to this problem:
  $\nabla^2 G(r,r') + k_0^2 G(r,r') = -\delta(r,r')$ [impulse resp] ②
  δ is a Dirac delta function

$$E(r) - E_i(r) = k_0^2 \int_D G(r,r')\,\chi(r')\,E(r')\,dr'$$

Recap: Solving the Integral Equation

$$E(r) - k_0^2 \int_D G(r, r')\,\chi(r')\,E(r')\,dr' = E_i(r)$$

- Discretize $E(r)$, $\chi(r)$ using "pulse" basis functions: $E(r) = \sum_{n=1}^{N} u_n p_n(r)$. The new variables are u_n.
- For each r location on the grid, we will get one equation in all N variables.
- Cycle through all the N locations to get a $N \times N$ system of equations.
- Solve to get all u_n and thus $E(r)$.

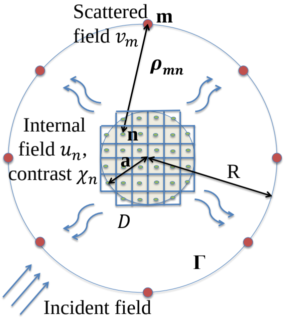


Table of Contents

Towards the inverse problem formulation

Our fav eqn: $E(r) - k_0^2 \int_D G(r, r')\, \chi(r')\, E(r')\, dr' = E_i(r)$

Towards the inverse problem formulation

Our fav eqn: $E(r) - k_0^2 \int_D G(r, r') \chi(r') E(r') \, dr' = E_i(r)$

Let's convert it to the language of linear algebra:

$E(r) \to u \quad [r \in D] \qquad \chi(r) \to x \qquad E_i(r) \to e$
Define scattered field as $E(r) - E_i(r) \to s \quad [r \notin D]$: all col vectors

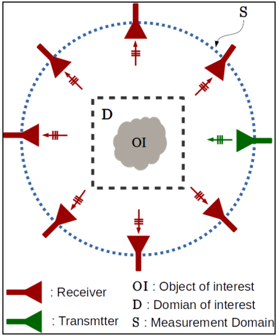
Towards the inverse problem formulation

Our fav eqn: $E(r) - k_0^2 \int_D G(r, r')\, \chi(r')\, E(r')\, dr' = E_i(r)$

Let's convert it to the language of linear algebra:

$E(r) \to u \quad [r \in D] \qquad \chi(r) \to x \qquad E_i(r) \to e$

Define scattered field as $E(r) - E_i(r) \to s \quad [r \notin D]$: all col vectors



Towards the inverse problem formulation

Our fav eqn: $E(r) - k_0^2 \int_D G(r, r')\, \chi(r')\, E(r')\, dr' = E_i(r)$

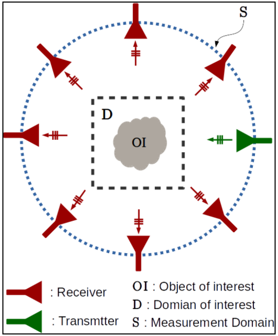
Let's convert it to the language of linear algebra:

$E(r) \to u \quad [r \in D] \qquad \chi(r) \to x \qquad E_i(r) \to e$

Define scattered field as $E(r) - E_i(r) \to s \quad [r \notin D]$: all col vectors

When $r \in D$

- $u - G_D X u = e$
- 'State' equation
- Can solve for u when X known
- G_D full rank: has unique soln



Towards the inverse problem formulation

Our fav eqn: $E(r) - k_0^2 \int_D G(r,r')\,\chi(r')\,E(r')\,dr' = E_i(r)$

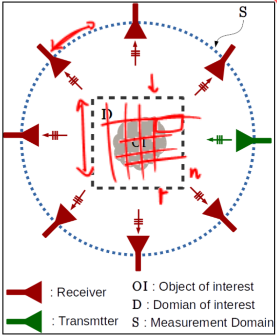
Let's convert it to the language of linear algebra:

$E(r) \rightarrow u \quad [r \in D] \qquad \chi(r) \rightarrow x \qquad E_i(r) \rightarrow e$

Define scattered field as $E(r) - E_i(r) \rightarrow s \quad [r \notin D]$: all col vectors

When $r \in D$

- $u - G_D X u = e$
- 'State' equation
- Can solve for u when X known
- G_D full rank: has unique soln



When $r \in S$

- $s = G_S U x$
- 'Data' equation
- Can solve for x when U known
- G_S under determined: no unique soln

More on the inverse problem – trouble lies ahead!

Let's delve more into the 'Data' equation,
connecting measurements s to desired parameter x

More on the inverse problem – trouble lies ahead!

Let's delve more into the 'Data' equation,
connecting measurements s to desired parameter x

- Typically s doesn't come by itself, some noise comes along for the ride! What we have is: s (actual) $= s$ (ideal) $+ \eta$ (noise)

More on the inverse problem – trouble lies ahead!

Let's delve more into the 'Data' equation,
connecting measurements s to desired parameter x

- Typically s doesn't come by itself, some noise comes along for the ride! What we have is: s (actual) $= s$ (ideal) $+ \eta$ (noise)
- So the problem to solve becomes: $\boxed{\hat{x} = \underset{x}{\text{argmin}} \|s - G_S U x\|_2}$

More on the inverse problem – trouble lies ahead!

Let's delve more into the 'Data' equation,
connecting measurements s to desired parameter x

- Typically s doesn't come by itself, some noise comes along for the ride! What we have is: s (actual) $= s$ (ideal) $+ \eta$ (noise)
- So the problem to solve becomes: $\boxed{\hat{x} = \underset{x}{\text{argmin}} \|s - G_S U x\|_2}$

Let's assume for a (magical) moment that U is known

- Linear algebra says that an underdetermined system has ∞ solutions
- I need some more information to constrain the solution, e.g. psuedo inverse soln (min 2-norm) or sparse solution (min 1-norm)

More on the inverse problem – trouble lies ahead!

Let's delve more into the 'Data' equation,
connecting measurements s to desired parameter x

- Typically s doesn't come by itself, some noise comes along for the ride! What we have is: s (actual) $= s$ (ideal) $+ \eta$ (noise)
- So the problem to solve becomes: $\hat{x} = \underset{x}{\operatorname{argmin}} \|s - G_S U x\|_2$

Let's assume for a (magical) moment that U is known

- Linear algebra says that an underdetermined system has ∞ solutions
- I need some more information to constrain the solution, e.g. psuedo inverse soln (min 2-norm) or sparse solution (min 1-norm)

meas

The problem now becomes: $\hat{x} = \underset{x}{\operatorname{argmin}}\{\|s - G_S U x\|_2 + \beta \|x\|_1\}$

Adding more info → 'Regularization'

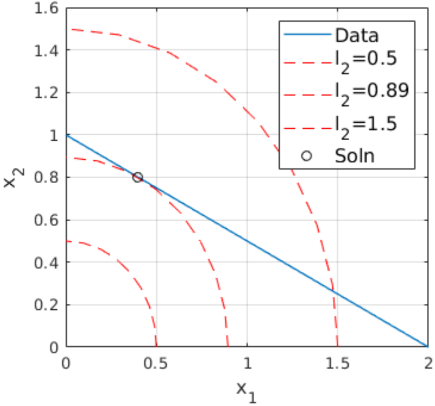
prediction

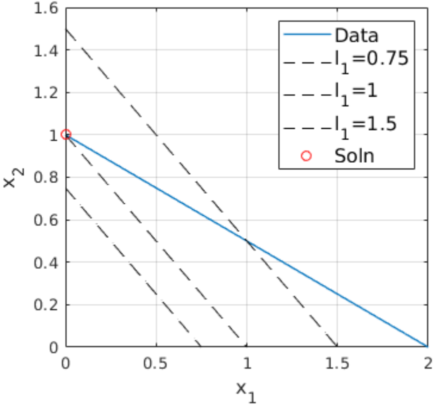
Aside: some kinds of regularization [1 data eqn, 2 vars]

Recall: dealing with under-determined system of equations $\implies$ ∞ solns

Aside: some kinds of regularization [1 data eqn, 2 vars]

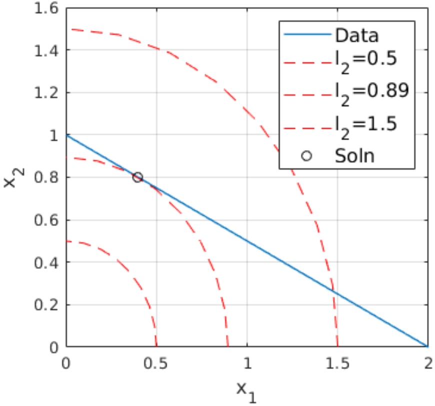
Recall: dealing with under-determined system of equations $\implies$ ∞ solns

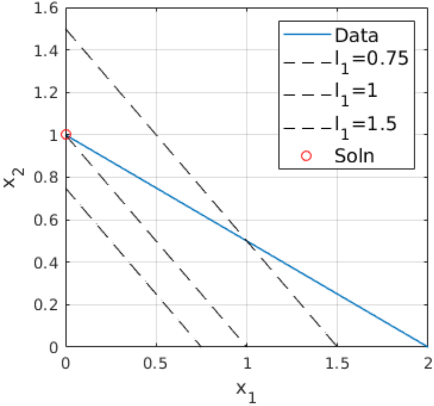




Aside: some kinds of regularization [1 data eqn, 2 vars]

Recall: dealing with under-determined system of equations $\implies$ ∞ solns



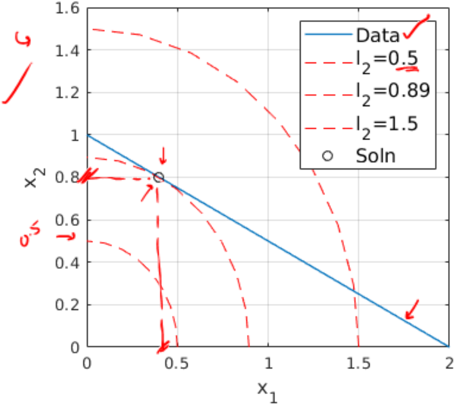


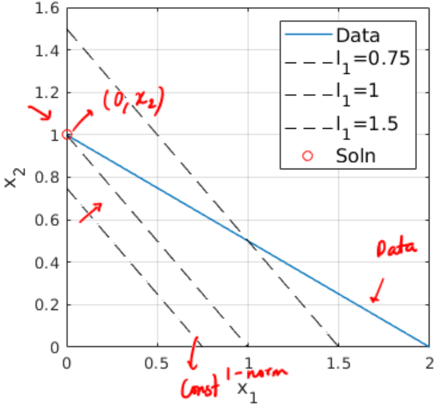
The solution with minimum ℓ_2 norm has all entries non-zero $\rightarrow$ soln is 'spread out' in all variables

Aside: some kinds of regularization [1 data eqn, 2 vars]

$\|x\|_1 = \sum_i |x_i|$

Recall: dealing with under-determined system of equations $\implies$ ∞ solns





The solution with minimum ℓ_2 norm has all entries non-zero $\rightarrow$ soln is 'spread out' in all variables

But solution with minimum ℓ_1 norm has some entries zero $\rightarrow$ soln is sparse in higher dims

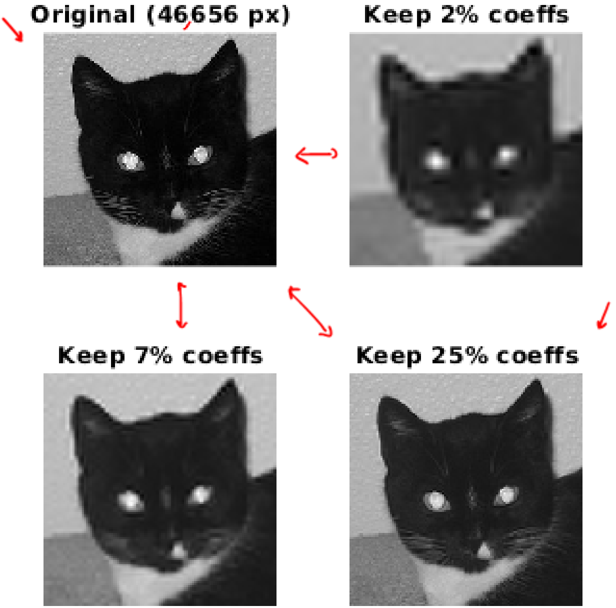
Why are *minimum* ℓ_1 norm solutions preferred?

Natural images are sparse in wavelet / discrete cosine basis.

Why are *minimum* ℓ_1 norm solutions preferred?

Natural images are sparse in wavelet / discrete cosine basis.

For example, in the Daubechies-4 Wavelet basis $\rightarrow$ reasonable reconstructions with few coefficients



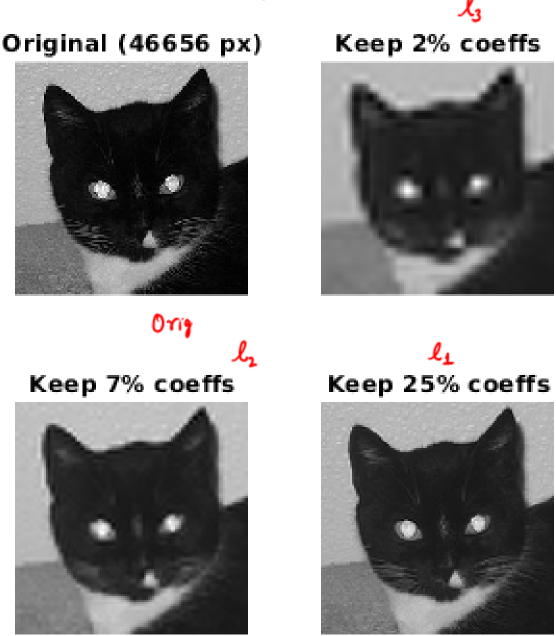
Why are *minimum* ℓ_1 norm solutions preferred?

Natural images are sparse in wavelet / discrete cosine basis.

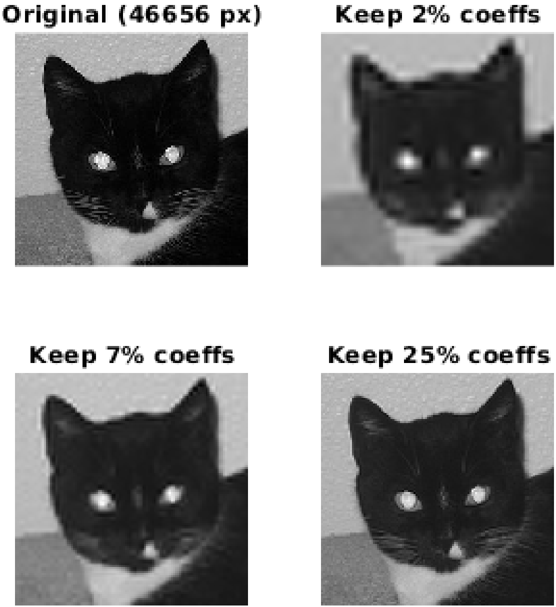
For example, in the Daubechies-4 Wavelet basis $\rightarrow$ reasonable reconstructions with few coefficients

Apriori knowledge of sparseness is a regularizer.

We don't need to know *which* coeffs are zero!



Why are *minimum* ℓ_1 norm solutions preferred?



These kind of solutions are studied in the field of **Compressive Sensing** – a new sub-field of Signal Processing since $\sim$2008

ill-posed

The inverse problem – More issues!

nonlinearity.

- In $\boxed{\underset{x}{\operatorname{argmin}}\{\|s - G_S U x\|_2 + R(x)\}}$ trouble is, U is not known.

The inverse problem – More issues!

- In $\boxed{\underset{x}{\text{argmin}}\{\|s - G_S U x\|_2 + R(x)\}}$ trouble is, U is not known.
- Why not use the 'State' eqn? $\boxed{u = (I - G_D X)^{-1} e}$
- Start with a guess for x, then alternate between solving the two:

$$\boxed{\rightarrow \quad \hat{x} = \underset{x}{\text{argmin}}\{\|s - G_S U x\|_2 + R(x)\}}$$

The inverse problem – More issues!

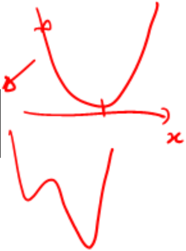
- In $\underset{x}{\operatorname{argmin}}\{\|s - G_S U x\|_2 + R(x)\}$ trouble is, U is not known.
- Why not use the 'State' eqn? $u = (I - G_D X)^{-1} e$
- Start with a guess for x, then alternate between solving the two:

$$\rightarrow \quad \hat{x} = \underset{x}{\operatorname{argmin}}\{\|s - G_S U x\|_2 + R(x)\}$$

Born Approx → $x = 0$.

- Above procedure called the Born Iterative Method
- OR, we can combine the two into one monster eqn:

$$\hat{x} = \underset{x}{\operatorname{argmin}}\{\|s - G_S \operatorname{diag}\left(((I - G_D X)^{-1} e\right) x\|_2 + R(x)\}$$



The inverse problem – More issues!

- In $\boxed{\underset{x}{\operatorname{argmin}}\{\|s - G_S U x\|_2 + R(x)\}}$ trouble is, U is not known.
- Why not use the 'State' eqn? $\boxed{u = (I - G_D X)^{-1} e}$
- Start with a guess for x, then alternate between solving the two:

$$\boxed{\rightarrow \quad \hat{x} = \underset{x}{\operatorname{argmin}}\{\|s - G_S U x\|_2 + R(x)\}}$$

- Above procedure called the Born Iterative Method
- OR, we can combine the two into one monster eqn:

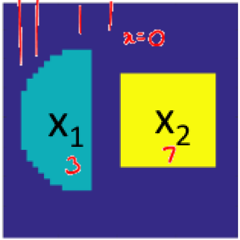
$$\boxed{\hat{x} = \underset{x}{\operatorname{argmin}}\{\|s - G_S \operatorname{diag}(((I - G_D X)^{-1} e) x\|_2 + R(x)\}}$$

What's the problem with this?

- ill-posed (not enough data)
- nonlinear (see above eqn)

An experiment to study nonlinearity

Consider a simple object to visualize the challenge

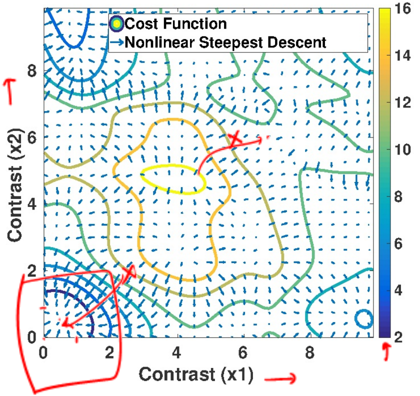


Since only two variables, we can visualize the maxima/minimas of this function

1. We will plot how $\|s - G_S U x\|_2$ looks like, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (exact U is also calculated at each x)
2. We can look at **linear** (assuming U const) and **nonlinear** approach (treating U as a fn of x)

Nonlinearity – Main Challenge, visualize $\|s - G_S U x\|_2$

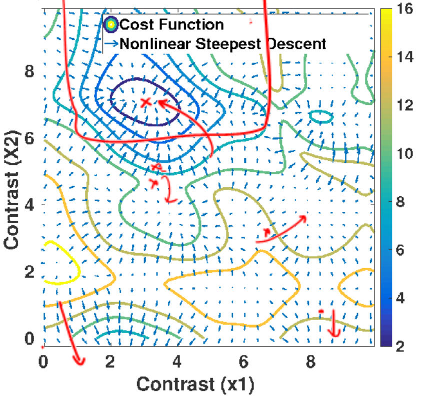
Low contrast object



Easy to arrive at the correct solution

$(x_1, x_2) = (0.5, 0.5)$

High contrast object



Many local minima along the way

$(x_1, x_2) = (3, 7)$

Table of Contents

Summary

- Integral equations are a powerful tool
- Forward problems $\rightarrow$

  unique solutions
- Inverse problems $\rightarrow$

  more interesting

  But,

  ill-posed

  nonlinear

Summary

- Integral equations are a powerful tool
- Forward problems → unique solutions
- Inverse problems → more interesting

But,

ill-posed ✓ not enough info

nonlinear → nonconvex optimization.

What skills do you need?

- Computational Electromagnetics
- Signal Processing ←
- Linear Algebra ✓
- Optimization
- Programming ✓

* (ML)