Computational Electromagnetics :
The 2D Vector Finite Element Method

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Topics in this module

1. Shape functions
2. Equation Setup, Converting to weak form, Boundary conditions
3. Choosing Variables
4. Putting it together

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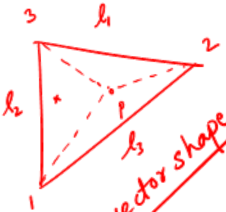
2D scalar Shape functions



$$L_i = \begin{cases} \dfrac{\text{Area }(P23)}{\text{Area }(123)} & P \in \Delta \\ 0 & P \text{ outside.} \end{cases}$$

<u>2D Node based</u>

$$L_i(x,y) = \frac{a_i + b_i x + c_i y}{2\Delta} \rightarrow U(x,y) = U_1 L_1(x,y) + U_2 L_2(x,y) + U_3 L_3(x,y)$$

Definition of vector shape fns

$$\vec{T}_k(x,y) = l_k \left(L_i \vec{\nabla} L_j - L_j \vec{\nabla} L_i \right) = \frac{l_k}{4\Delta^2} \left\{ \begin{array}{l} (A_k + B_k y, \\ C_k + D_k x) \end{array} \right\}$$

l_k: length of k^{th} edge

$\vec{T}_1, \vec{T}_2, \vec{T}_3$

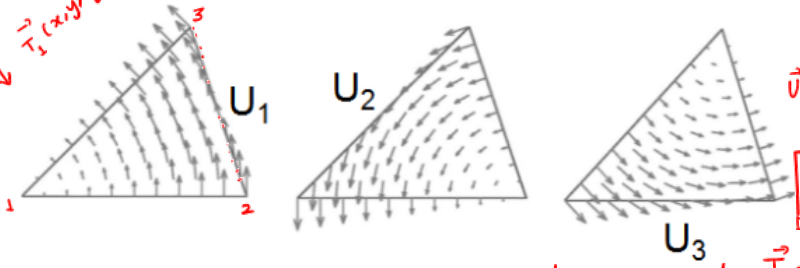
$$T_1 = l_1 (L_2 \nabla L_3 - L_3 \nabla L_2)$$

- Whitney elements
  - Nedelec elements.

$\vec{T}_2(x,y)$ → all zero outside △

2D vector Shape functions

$\vec{T}_1(x,y)$



$\vec{T}_3(x,y)$.

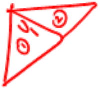
Field at any pt inside:

$$\vec{U}(x,y) = U_1\vec{T}_1(x,y) + U_2\vec{T}_2(x,y) + U_3\vec{T}_3(x,y)$$

$\boxed{\text{Unknowns are scalars } U_1, U_2, U_3.}$

1) $\vec{T}_1$ along edge 2-3 : constant component $\vec{T}_1 \cdot \hat{r}_{2-3} = \pm 1$ (Not inside)

2) $\vec{T}_1$ along edges 1-2, 3-1 : Normal to other edges $\vec{T}_1 \cdot \hat{r}_{1-3} = 0$, $\vec{T}_1 \cdot \hat{r}_{1-2} = 0$



Tangential boundary condns
→ Automatically satisfied.

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Vector Wave Equation $\to$ Weak Form

$\nabla \times \vec{H} = j\omega\varepsilon \vec{E}$, $\nabla \times \vec{E} = -j\omega\mu \vec{H}$ $\qquad (\varepsilon = \varepsilon_r \varepsilon_0)$

2D, TM pol.

$\frac{1}{\varepsilon_r(r)} \nabla \times \vec{H} = j\omega\varepsilon_0 \vec{E} \; \cancel{+\vec{J}}$

$\nabla \times \left[\frac{1}{\varepsilon_r} \nabla \times \vec{H} \right] = j\omega\varepsilon_0 (\nabla \times \vec{E})$

$= \omega^2 \varepsilon_0 \mu_0 \mu_r(r) \vec{H}$

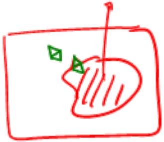
$\vec{F}_H = \nabla \times \left[\frac{1}{\varepsilon_r} \nabla \times \vec{H} \right] - k_0^2 \mu_r \vec{H} = 0$

vector wave Eqn.

Ideally $\vec{F}_H(r) = 0 \;\; \forall r$

Instead, FEM say:

$\int_\Omega \vec{T}_m(r) \cdot \vec{F}_H(r) \, d\vec{r} = 0$ $\qquad d\vec{r} \to dx\,dy$



Weighted Residual Method.

$\iint \vec{T}_m(r) \cdot \left[\nabla \times \frac{1}{\varepsilon_r(r)} (\nabla \times \vec{H}(r)) - k_0^2 \mu_r \vec{H} \right] d\vec{r} = 0$

$\iint_\Omega \nabla \cdot \vec{F} \, ds = \oint_C \vec{F} \cdot \hat{n} \, dl$

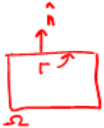
$\vec{A} \cdot (\nabla \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \nabla \cdot (\vec{A} \times \vec{B})$

2D DIV thm

1st term:

$\iint \left[(\nabla \times \bar{T}_m) \cdot \left(\frac{1}{\varepsilon_r} \nabla \times \bar{H} \right) - \nabla \cdot \left(\bar{T}_m \times \left(\frac{1}{\varepsilon_r} \nabla \times \bar{H} \right) \right) \right] d\vec{r}$

$\quad '' \quad '' \quad - \oint_\Gamma \bar{T}_m \times \left(\frac{1}{\varepsilon_r} \nabla \times \bar{H} \right) \cdot \hat{n} \, dl$



Weak form:

$\iint_\Omega \left[(\nabla \times \bar{T}_m) \cdot \left(\frac{1}{\varepsilon_r} \nabla \times \bar{H} \right) - k_0^2 \mu_r \bar{T}_m \cdot \bar{H} \right] d\vec{r}$

$= \oint_\Gamma \vec{T}_m \times \frac{1}{\varepsilon_r} (\nabla \times \vec{H}) \cdot \hat{n} \, dl.$

$= \oint \vec{T}_m \cdot \hat{n} \times \frac{1}{\varepsilon_r} (\nabla \times \vec{H}) \, dl$

RHS : $\oint_\Gamma - \oint_{\Gamma'} (\;)$

$\nabla^2 \vec{E} + k^2 \vec{E} = 0$, $\vec{E} = E_0 \exp(-j\vec{k}\cdot\vec{r})$ ↑ a soln

$$\vec{E} = \int_{-\infty}^{\infty} E_0(p)\, \exp(-j\,\vec{k}\cdot\vec{r})\, dp \leftarrow \text{also soln}$$

(k_x, k_y, k_z) soln.

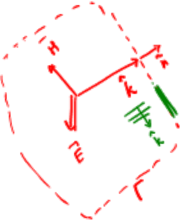
Any wave: collection of plane waves.

$\nabla\times\vec{H} = j\omega\varepsilon\vec{E}$: $\vec{H} : \vec{H}_0\, e^{-j\vec{k}\cdot\vec{r}}$

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ $\quad -jk_x, -jk_y, -jk_z$

$$\nabla\times\vec{H} = -j\vec{k}\times\vec{H} = -jk(\hat{k}\times\vec{H})$$

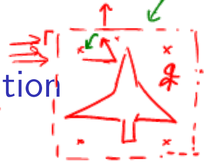
An expression satisfied by a plane wave



$$\underbrace{\hat{n}\times\frac{1}{\varepsilon_r}(\nabla\times\vec{H})} = \hat{n}\times\frac{1}{\varepsilon_r}\left(-jk\,\hat{n}\times\vec{H}\right) = \frac{-jk}{\varepsilon_r}\left[\hat{n}\times(\hat{n}\times\vec{H})\right]$$

True for a plane wave hitting Γ normally.

Radiation Boundary Condition



or 1st order absorbing B.C.

Weak form RHS

$$\oint_{\Gamma} \vec{T}_m\cdot\hat{n}\times\frac{1}{\varepsilon_r}(\nabla\times H)\,dl \quad [\text{earlier, exact}]$$

$$\oint_{\Gamma} \vec{T}_m\cdot\left[\hat{n}\times(\hat{n}\times\vec{H})\right]\left(\frac{-jk}{\varepsilon_r}\right)dl \quad [\text{approx}]$$

× Not correct when $\hat{k} \neq \hat{n}$

× Leads to numerical reflections

$\Rightarrow$ Larger comp domain.

× Obeyed by only Scattered fields

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Total field formulation

Variable of interest = Total field

$\vec{H}_{tot} = \vec{H}_{sc} + \vec{H}_{in} \longrightarrow$ known

↳ easily satisfy 'natural' Maxwell's tangential B.C.'s.

↳ slightly harder to impose RBC.

Only on boundary. in $Ax = b$

$$\left[\hat{n} \times \left(\frac{1}{\varepsilon_r} \nabla \times \vec{H}\right)\right] \to \hat{n} \times \frac{1}{\varepsilon_r}\left(\nabla \times \left[\vec{H}_{inc} + \underbrace{\vec{H} - \vec{H}_{in}}_{\vec{H}_{sc}}\right]\right)$$

$$= \hat{n} \times \frac{1}{\varepsilon_r}\left(\nabla \times \vec{H}_{in}\right) + \hat{n} \times \frac{1}{\varepsilon_r} \nabla \times (\vec{H} - \vec{H}_{in})$$

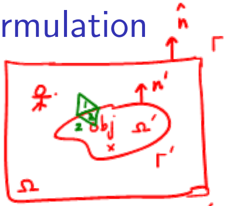
$$= \quad " \quad " \quad - \frac{jk}{\varepsilon_r} \hat{n} \times (\hat{n} \times (\vec{H} - \vec{H}_{in}))$$

$$= \underbrace{\hat{n} \times \frac{1}{\varepsilon_r} \nabla \times \vec{H}_{in} + \frac{jk}{\varepsilon_r} \hat{n} \times (\hat{n} \times \vec{H}_{in})}_{\text{known}} - \underbrace{\frac{jk}{\varepsilon_r} \hat{n} \times (\hat{n} \times \vec{H})}_{\text{unknown}}$$

Variable of interest is

$\vec{H}_{tot} = \underbrace{\vec{H}_{sc}}_{\text{unknown}} + \vec{H}_{in}$

Scattered field formulation



Tot Domain : $\Omega \cup \Omega'$

$\Omega \cap \Omega' = \phi$

Start with Ω : (variable H_s)

$$\iint_{\Omega} \left((\nabla \times \vec{T}) \cdot \frac{1}{\epsilon_r} (\nabla \times \vec{H}_s) - k_0^2 \mu_r \vec{T} \cdot \vec{H}_s \right) d\bar{r} = \left(\oint_{\Gamma} - \oint_{\Gamma'} \right) \left[\vec{T} \times \left(\frac{1}{\epsilon_r} \nabla \times \vec{H}_s \right) \cdot \hat{n} \right] dl$$

$$= \oint_{\Gamma} (\ ) - \oint_{\Gamma'} (\ ) \quad -(1)$$

About Ω' (variable H)

$$\iint_{\Omega} \left((\nabla \times \vec{T}) \cdot \frac{1}{\epsilon_r} (\nabla \times \bar{H}) - k_0^2 \mu_r \vec{T} \cdot \bar{H} \right) d\bar{r} = \oint_{\Gamma'} \left[\vec{T} \times \left(\frac{1}{\epsilon_r} \nabla \times H \right) \cdot \hat{n} \right] dl \quad -(2)$$

Eq(1) RHS : term (1) $\rightarrow$ apply RBC $= -j \frac{k}{\epsilon_r} \left[\hat{n} \times (\hat{n} \times H_s) \right]$

term (2) & RHS of eqn(2) $\rightarrow$ leave as is.

$\vec{H} - \vec{H}_s = \vec{H}_i \quad -(3)$

Tr(2) $\rightarrow \vec{H}$

Tr(1) $\rightarrow \vec{H}_s$

unknown

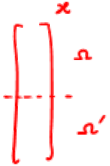
$\frac{d^2}{dx^2} E + k_0^2 E = 0$ (vac)

$\hookrightarrow E = E_0 e^{-jk_0 x}$?

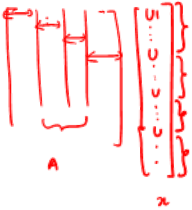
$\frac{d^2 E}{dx^2} + k^2 E = 0$ (obj)

$-k_0^2 + k^2 \neq 0$

$\Rightarrow$ Inc field doesn't obey



Comparing Total (TF) & Scattered field (SF) formulation



→ $\in \Omega, \notin \Gamma'$
→ $\in \Omega, \in \Gamma'$ } Scat

→ $\in \Omega', \in \Gamma'$
→ $\in \Omega', \notin \Gamma'$ } total

$H_s - H = -H_i$

$$\begin{bmatrix} \underline{I} & -\underline{I} \end{bmatrix} \begin{bmatrix} H_s \\ H \end{bmatrix} = \begin{bmatrix} -H_i \end{bmatrix}$$

set 2

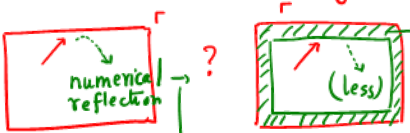
set 3

Inc field

↳ TF: appeared at Γ ✓

SF: appeared at Γ'

grid dispersion



numerical reflection → ?

due to inexact BC

Absorber

ε_r → make it lossy.

One way of improving BC.

Absorbers:

↳ TF: More errors due to prop thru abs.

↳ SF: No change.

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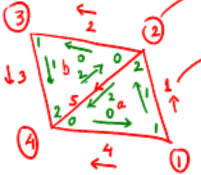
Assembling the system of equations



Global Node numbers

Global edge nos

Conv: Edge points from smaller # to larger no #.

$$\Phi(\vec{T},\vec{H}) = (\nabla\times\vec{T})\cdot\frac{1}{\epsilon_r}(\nabla\times\vec{H}) - \mu_r k_0^2\, \vec{T}\cdot\vec{H}$$

LHS term

Testing along edge #5 : Non-zero over #a, #b : $\underbrace{\overbrace{\vec{T}_2^{a}(\vec{r})} + \overbrace{\vec{T}_2^{b}(\vec{r})}}_{\vec{T}}$ ← Testing fn.

el a → U_1, U_4, U_5

el b → U_2, U_3, U_5

$$\iint_\Omega \Phi(\vec{T},\vec{H})\,ds = \iint_a + \iint_b .$$

$\boxed{\vec{H} = \sum U_i \vec{T}_i}$ (expanding $\vec{H}$ in basis fns)

$$\left.\begin{aligned} a:&\quad \vec{H} = U_1^a \vec{T}_1^a + U_0^a \vec{T}_0^a + U_2^a \vec{T}_2^a \\ b:&\quad \vec{H} = U_0^b \vec{T}_0^b + U_1^b \vec{T}_1^b + U_2^b \vec{T}_2^b \end{aligned}\right\} \text{local } U\text{'s}$$

$$\left.\begin{aligned} a:&\quad \vec{H} = U_1 \vec{T}_1^a - U_4 \vec{T}_0^a + U_5 \vec{T}_2^a \\ b:&\quad \vec{H} = U_2 T_0^b + U_3 T_1^b - U_5 T_2^b \end{aligned}\right\} \text{global } U\text{'s}$$

$$= \iint_a \left[U_1 \Phi(\vec{T}_2^a, \vec{T}_1^a) - U_4 \Phi(T_2^a, T_0^a) + U_5 \Phi(T_2^a, T_2^a)\right] ds$$

$$+ \iint_b \left[U_2 \Phi(T_2^b, T_0^b) + U_3 \Phi(T_2^b, T_1^b) - U_5 \Phi(T_2^b, T_2^b)\right] ds$$

$$\left[A_{2,1}^a,\ A_{2,0}^b,\ A_{2,1}^b,\ -A_{2,0}^a,\ A_{2,2}^a - A_{2,2}^b\right] \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = b_5$$

2D TM ($\vec{H}_{x-y}, E_z$)

Computing the far-field

× Brute force

✓ Huygen's principle

$$E_z^{sc}(\vec{r}) = \oint_R \Big[\underbrace{E_z(\vec{r}\,')}_{\text{Numerically}}\nabla g(\vec{r},\vec{r}\,')\cdot\hat{n} - \underbrace{\nabla E_z(\vec{r}\,')\cdot\hat{n}}_{\propto\,\vec{H}_{tan}}\, g(\vec{r},\vec{r}\,')\Big]\, dl'$$

$\vec{H}_{tan}$ directly computed.

indep of r for r>>1

$$H_0^{(2)}(k\rho) \approx \sqrt{\frac{2j}{\pi k \rho}}\, e^{-jk\rho} \quad \text{(far field)}$$

$$\sigma_{2D} = \lim_{r\to\infty}\left(2\pi r \left|\frac{E_z^s(r)}{E_z^i(r)}\right|^2\right)$$

Radar Cross-section.

$\rho >> 1$

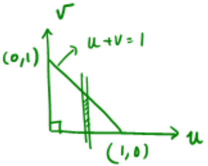
$$\begin{cases} \nabla\times\bar{T} = \text{const} \\ \nabla\cdot\bar{T} = 0 \end{cases}$$

$$\Phi(\vec T,\vec H) \to \Phi(\vec T_m,\vec T_n) = \Big[\underbrace{(\nabla\times\vec T_m)\cdot\frac{1}{\underline{\epsilon_r}}(\nabla\times\vec T_n)} - \underline{k_0^2}\,\underline{\mu_r}\,\underbrace{\vec T_m\cdot\vec T_n}\Big] \leftarrow \quad \begin{vmatrix} \hat x & \hat y & \hat z \\ \partial/\partial x & \partial/\partial y & \partial z \\ A_k+B_k y & C_k+D_k x & 0 \end{vmatrix}$$

Numerical aspects in computing matrix elements

$$\vec T_k = \frac{l_k}{4\Delta^2}\left(A_k+B_k y,\ C_k+D_k x\right) \quad \sim \nabla\times\vec T = \text{const.} \quad \vec T_m\cdot\vec T_n \to \text{quadratic in } x,y.$$

We want : $\displaystyle\iint_e \underbrace{\Phi(\vec T_m,\vec T_n)}\,dx\,dy.$



?

$$\iint_\Delta f(u,v)\,du\,dv = \int_{u=0}^{1}\left[\int_{v=0}^{1-u} f(u,v)\,dv\right]du$$

$$\iint_\Delta u^q v^p\,du\,dv = \frac{p!\,q!}{(p+q+2)!}$$

$$x = x_1 + \overset{a}{(x_2-x_1)}\,u + \overset{b}{(x_3-x_1)}\,v$$

$$y = y_1 + \underset{c}{(y_2-y_1)}\,u + \underset{d}{(y_3-y_1)}\,v$$

Affine transformation.

$$J = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$$\iint_e \underline{x\,dx\,dy} = \iint_\Delta (x_1 + a\underline{u} + b\underline{v})\,J\,du\,dv$$

$$\iint_e f(x,y)\,dx\,dy \to \iint_\Delta \tilde f(u,v)\,J\,du\,dv$$

J: Jacobian

RCS ?

Summary of FEM procedure

pre-processing
1) Generate a mesh file, store it (CAD s/w)
2) Read in the mesh file
3) Creating data structures
   (global ↔ local mappings)
   (list of elements ↔ nodes/edges)

processing
4) Matrix Assembly. (i.e. forming $Ax=b$)
5) Solving matrix
   → Direct methods (LU)
   → Iterative methods
      ↳ e.g. CG methods

post-processing
6) Compute quantity of interest, eg. RCS.
7) Visualizing output

Validate against known soln
→ Fresnel reflect coeffs.
→ Mie series solns — cylinder/sphere.

— x —

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References: Ch 4 of FEM for Electromagnetics; Volakis, Chatterjee, Kempel; IEEE Press

Instructor notes on 2D edge based FEM:

http://www.ee.iitm.ac.in/uday/notes/fem2dprimer.pdf