Computational Electromagnetics : The 1D Finite Element Method

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Topics in this module

1. Equation Setup
2. Converting to weak form
3. Discretization & Solution

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A generic differential equation

$$-\frac{d}{dx}\left(p(x)\frac{dU}{dx}\right)+q(x)U(x)=f(x), \quad 0<x<x_a$$

unknowns: $U(x)$, knowns: $p(x), q(x), f(x)$

① Parallel plate capacitor. $\nabla^2 V = -\rho/\varepsilon$

$U(x)$: potential, $U(0)=0$, $U(x_a)=V_0$

$p(x)=-1$, $q(x)=0$, $f(x)=-\rho/\varepsilon$

$$\frac{d}{dx}\left(\frac{dU}{dx}\right)=-\rho/\varepsilon$$

② wave between parallel plates.

$E''+k_0^2E=\alpha J$

$p(x)=-\frac{1}{\mu_r}$, $q(x)=k_0^2\varepsilon_r$

$$\frac{d}{dx}\left(\frac{1}{\mu_r}\frac{dE}{dx}\right)+k_0^2\varepsilon_r E=f(x)$$

FEM ↪ Weighted Residual Method & Requirements on $W_m(x)$

$$R(x) = -\frac{d}{dx}\left(p(x)\frac{dU}{dx}\right) + q(x)U(x) - f(x)$$

$$\int_{\Omega_m} \boxed{W_m(x)} \cdot R(x)\, dx = 0$$

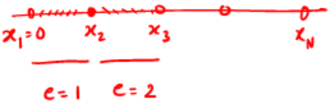
weight fn or
basis fn. or
testing fn or
trial fn

unknowns U_2 → global name for unknown

→ local names: $U_2^{e=1} = U_1^{e=2} = U_{\boxed{2}}$

local name: L or R

global number



Mapping between local & global should be maintained.

Requirements: ① $W_m(x)$ & its derivative square integrable over the domain.

$$\Rightarrow \int_0^{x_a} \left\{ \left(W_m(x)\right)^2 + \left(\frac{d}{dx}W_m(x)\right)^2 \right\} dx < \infty$$

② $W_m(x)$ must obey boundary conds.

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Towards a system of equations - 2 Steps

Integration by parts.

Step 1:

$$\int_{\Omega_m} W_m(x)\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}U(x)\right)\right]dx = \Big[W_m(x)\,p(x)\frac{d}{dx}U(x)\Big]_{\text{end pts}} - \int_{\Omega_m}\left(\frac{d}{dx}W_m(x)\right)p(x)\frac{d}{dx}U(x)\,dx$$

2nd, 1st

known

$\Sigma\Sigma U_i^e N_i^e(x)$, $N_i^e(x)$

Step 2:

$$\int_{\Omega_m}\left[\left(\frac{d}{dx}W_m(x)\right)p(x)\frac{d}{dx}U(x) + W_m(x)q(x)U(x) - W_m(x)f(x)\right]dx - \Big[W_m(x)\,p(x)\frac{d}{dx}U(x)\Big]_{\text{end pts.}} = 0$$

1) Weak form of FEM.

2) B.C.'s appear in this eqn.

$$\int_{\Omega_m} W_m(x)R(x)\,dx = 0$$

$$R(x) = \left[-\frac{d}{dx}\left(p(x)\frac{d}{dx}U(x)\right) + q(x)U(x) - f(x)\right]$$

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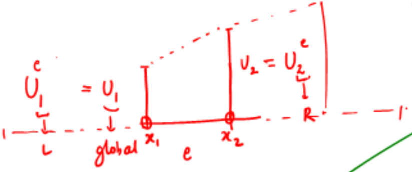
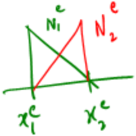
Discretization and shape/basis functions

tesselation of domain

Breaking into elements (segments)



$$N_1^e = \begin{cases} \dfrac{x_2^e - x}{x_2^e - x_1^e} & x_1^e \le x \le x_2^e \\ 0 & \text{else} \end{cases}, \quad N_2^e = \begin{cases} \dfrac{x - x_1^e}{x_2^e - x_1^e} & x_1^e \le x \le x_2^e \\ 0 & \text{else} \end{cases}$$



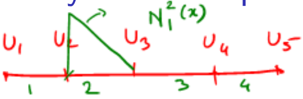
$$U(x) = \sum_{e=1}^{N_e} \left[U_1^e N_1^e(x) + U_2^e N_2^e(x) \right] = \sum_{e=1}^{N_e} \sum_{i=1}^{2} \left(U_i^e N_i^e(x) \right)$$

Substitute into the weak form

Generating the system of equations

unknown known

$$U(x) = \sum_{e=1}^{N_e} \sum_{i=1}^{2} U_i^e \, N_i^e(x)$$



$$\underline{W_m(x)} = \underline{N_i^e(x)} \leftrightarrow \underbrace{\text{L.C.'s of } N_i^e(x)\,?}$$

$$\sum_{e=1}^{N_e} \sum_{i=1}^{2} U_i^e \int_e \left[p(x) \frac{dW_m(x)}{dx} \frac{dN_i^e(x)}{dx} + \underline{q(x)\, W_m(x)\, N_i^e(x)} \right] dx - \int W_m(x) f(x)\, dx - \underline{\left[p(x)\, W_m(x) \frac{dU}{dx} \right]_{x=x_1^e}^{x=x_2^e}} = 0$$

Weak form

↳ Substitute $W_m(x)$ for various values of m.

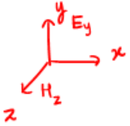
→ Get a system of equations.

→ Sparse? Choose $\underline{W_m(x)} = \underline{N_{i=1}^{e=2}(x)} \rightarrow U_2, U_3$

↘ Overall sparse system.

Example problem: 1D wave equation

$$\nabla^2 U + k^2 u = 0 \rightarrow \frac{d^2 U(x)}{dx^2} + k^2 U(x) = 0$$



$$\vec{E} = U(x)\,\hat{y}, \quad \vec{H} = \frac{U(x)}{\eta}\,\hat{z}, \quad \nabla \times \vec{E} = -j\omega\mu\vec{H} = \hat{z}\,\frac{\partial U(x)}{\partial x} = -j\omega\mu\frac{U(x)}{\eta}\,\hat{z}$$

$$\Rightarrow \frac{dU(x)}{dx} = \frac{-j\omega\mu}{\eta}U(x)$$

Dirichlet: $U(x_0) = \text{const}$

Neumann: $\left.\frac{d}{dx}U(x)\right|_{x=x_0} = \text{const}$

Boundary condn

$$\left.\underbrace{\frac{d}{dx}U(x)}\, + \underbrace{j\frac{\omega\mu}{\eta}U(x)}\, = 0\right|_{\text{at boundary}}$$

$\alpha u' + \beta u = \text{const}$

→ Robin boundary condn.
Impedance " "
Sommerfield / Radiation b.c.

1) $\int_{\Omega_w} w(x)\left(u''(x) + k^2 u(x)\right)dx = 0$ ← Weighted Residual

2) $\left. w(x)u'(x) \right]_{\text{end pts}} - \int_{\Omega_w} w'(x)u'(x)\,dx + \int_{\Omega_w} k^2 w(x)\,u(x)\,dx = 0$

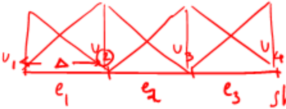
Weak form

Example problem: 1D wave equation

unknowns $\longrightarrow U_i^e \quad e = 1,2,3, \; i = 1,2$

$\hookrightarrow u(x) = \sum\sum U_i^e N_i^e(x) = U_1 N_1 + U_2 T_2 + U_3 T_3 + U_4 N_2$ local

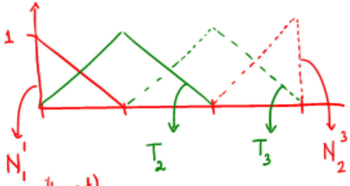
6 shape fns



shape fns $\quad u(x) = U_1^1 N_1^1(x) + U_2^1 N_2^1(x) + U_1^2 N_1^2(x) + U_2^2 N_2^2(x) + U_1^3 N_1^3(x) + U_2^3 N_2^3(x)$

$= U_1 N_1^1(x) + U_2 \left[N_2^1(x) + N_1^2(x) \right] + U_3 \left[N_2^2(x) + N_1^3(x) \right] + U_4 N_2^3(x)$

global



testing / weight fns. $\quad -\int w'(x)\, u'(x)\, dx + \int k^2 w(x)\, u(x)\, dx = -\left[w(x)\, u'(x) \right]$ endpts.

fn of x.

$U_{in} = e^{j(kx - \omega t)}$

testing with : $W_1(x) = N_1^1(x) \to N_1$

$w_1'(x) = \frac{-1}{\Delta}$

$-\int_{e_1} \underbrace{\left(\frac{-1}{\Delta}\right)\left[-\frac{U_1}{\Delta} + \frac{U_2}{\Delta} \right]}_{\text{const}} dx + \int_{e_1} \underbrace{k^2 \left[N_1^1(x)\right]\left[U_1 N_1^1(x) + U_2 N_2^1(x) \right]}_{\text{quadrature.}} dx$

$= -\left[0 - (\;) \right]$ node 2

$= \alpha U_1 + U_{in}'(x)\Big|_{x=x_1} - \alpha U_{in}(x)\Big|_{x=x_1}$

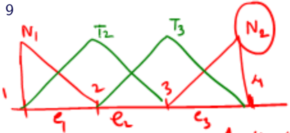
$E_s'(x) = \alpha E_s(x)$

$U - U_{in} = E_s$

$\frac{d}{dx}\left[U(x) - U_{in}(x) \right] = \alpha \left[U(x) - U_{in}(x) \right]$

$\Rightarrow \left[\frac{d}{dx} U(x) = \alpha U(x) + \frac{d}{dx} U_{in}(x) - \alpha U_{in}(x) \right]$

$E_t = E_s + E_i$



Example problem: 1D wave equation

$$-\underbrace{\int w'(x)\, U'(x)\, dx} + \underbrace{\int k^2 w(x)\, U(x)\, dx} = -\underbrace{\Big[w(x)\, U'(x) \Big]_{\text{end pts.}}}$$

1st eqn: $A_{11} U_1 + A_{12} U_2 + 0 + 0 = b_1$

testing with $w_2(x) = T_2(x)$.

$$-\int_{e_1} \frac{1}{\Delta}\left[-\frac{U_1}{\Delta} + \frac{U_2}{\Delta}\right] dx - \int_{e_2} \frac{-1}{\Delta}\left[-\frac{U_2}{\Delta} + \frac{U_3}{\Delta}\right] dx$$

$$\int_{e_1} k^2 \Big[N_2'(x)\big(U_1 N_1'(x) + U_2 N_2'(x)\big)\Big] dx + \int_{e_2} k^2 \Big[N_1^2(x)\big(U_2 N_1^2(x) + U_3 N_2^2(x)\big)\Big] dx$$

$$-\Big[T_2(x)\, U'(x) \Big]_{\text{node 1}}^{\text{node 3}} = 0$$

2nd eqn: $A_{21} U_1 + A_{22} U_2 + A_{23} U_3 + 0 = 0$

3rd eqn: $0 + A_{32} U_2 + A_{33} U_3 + A_{34} U_4 = 0$

4th eq: $0 + 0 + A_{43} U_3 + A_{44} U_4 = b_4$

$AU = b$

b → inc field.

$O(d\, n) \approx O(n)$

d → no of off diagonal els.

Topics that were covered in this module

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Reference: Ch 3 of FEM for Electromagnetics; Volakis, Chatterjee, Kempel; IEEE Press