Computational Electromagnetics: Method of Moments

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Topics in this module

1. Motivation
2. Linear Vector Spaces
3. Formulating the Method of Moments
4. MoM: Surface Integral Equations
5. MoM: Volume Integral Equations
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2 Linear Vector Spaces

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4 MoM: Surface Integral Equations

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Recall potential problem (finding $V(r) \forall r$)

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_L \frac{\rho(r')}{R} \, dl'$$

Steps: (1) Find $\rho$, (2) then $V$.

Problems with this approach?

- Need large $N$.
- Basis functions can be better.
- Not enforcing $V(y) = V_0$ at pts other than $y_m$.

Want a more robust method.
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From continuous to the discrete world

Integral/differential operators $\rightarrow$ Linear operator $\mathbb{L} \phi(r) = f(r)$

(a) Need a basis for $D(\mathbb{L})$

(b) Need a basis for $R(\mathbb{L})$

Condition on $f$? $f$ must be in $R(\mathbb{L})$

Once we discretize $\rightarrow$

finite dimensional vector space $V \rightarrow V_N$

Characterized by:

$\{ b_n, n = 1, \ldots, N \}$

1) Linearly independent
2) Span the vector space

basis
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Formulating a system of equations

Express $\phi(r)$ in the basis $\{b_n(r)\}_{n=1}^N$

$$\phi(r) = \sum_{n=1}^N \phi_n b_n(r)$$

$$(b_n(r), \phi(r)) = \phi_m$$

Unknowns now are: $\phi_n, t_n(r), f_n$

Choosing one $t_m(r)$ ‘testing’ fn gives:

$$(t_m(r), f(r)) \rightarrow \sum_{n=1}^N \phi_n (t_m(r), L b_n(r)) = f_m = (t_m(r), f(r))$$

Overall matrix equation becomes:

$$A x = c$$

$$A_{mn} = (t_m(r), L b_n(r))$$

$$c_m = f_m$$

Similarly for $f(r)$ in basis $\{t_n(r)\}_{n=1}^N$

$$f(r) = \sum_{n=1}^N f_n t_n(r)$$

Boundary condn? e.g. $L = \frac{d}{dx}$ & $L \phi(r) = 1$
Old wine in new bottle

In the first problem of \( \frac{1}{4\pi \epsilon_0} \int_L \frac{\rho(r')}{R} \, dl' = V(r'), \)
how to describe the \textbf{old} solution procedure in the \textbf{new} language?

1) \textbf{basis:} \( \rho(r) = \sum_{n} p_n b_n(r) \)

2) \textbf{testing:} \( V(r_m) = \int_0 V(r) \delta(r-r_m) \, dr \)

\[ \Rightarrow \ t_m(r) = \delta(r-r_m) \]

\textbf{pulse basis, delta testing}

\textbf{point testing}

\textbf{point collocation method.}
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Surface Integral Equations: Recap

\[ \phi_i(r') - \oint [ g_1(r, r') \nabla \phi_1(r) - \phi_1(r) \nabla g_1(r, r') ] \cdot \hat{n} dl \]

\[
= \begin{cases} 
\phi_1(r') & r' \in V_1 \\
0 & r' \in V_2 
\end{cases}
\]

Similarly for region 2:

\[ \oint [ g_2(r, r') \nabla \phi_2(r) - \phi_2(r) \nabla g_2(r, r') ] \cdot \hat{n} dl \]

\[
= \begin{cases} 
\phi_2(r') & r' \in V_2 \\
0 & r' \in V_1 
\end{cases}
\]
Surface Integral Equations: Recap (contd.)

Use only the Extinction theorem:

\[
\oint \left[ g_1(r, r') \nabla \phi(r) \cdot \hat{n} - \phi(r) \nabla g_1(r, r') \cdot \hat{n} \right] \, dl = \phi_i(r'), \quad r' \in V_2
\]

\[
\oint \left[ g_2(r, r') \nabla \phi(r) \cdot \hat{n} - \phi(r) \nabla g_2(r, r') \cdot \hat{n} \right] \, dl = 0, \quad r' \in V_1
\]

\[\nabla \phi \cdot \hat{n} = \sum_{n=1}^{N} a_n p_n(r) \quad \phi(r) = \sum_{n=1}^{N} b_n p_n(r) \quad \text{for } r \in V_1\]

Boundary Integral method.

Extended BC method.
Surface Integrals: Which terms are problematic?

\[ g(r, r') = - \frac{j}{4} H_0^{(2)}(k|r - r'|) \]

What about \( \nabla g \)? Use \( \frac{dH_0^{(2)}(x)}{dx} = -H_1^{(2)}(x) \)

Call \( \rho = |r - r'| = \sqrt{(x - x')^2 + (y - y')^2} \)

For \( \rho \ll 1 \):

\[ H_0^{(2)}(k\rho) \approx 1 - \frac{j}{2} \left( \ln \frac{k\rho}{2} + \gamma \right) \approx 0.57 \]

Both \( g \) and \( \nabla g \) blow up as \( \rho \to 0 \)

Thus, care while integration:

- Segments where \( r \neq r' \) \rightarrow Numerical quadrature rules
- Segments where \( r = r' \) \rightarrow Singular integrals
Surface Integrals: Kinds of singularities?

\[ H_0^{(2)}(x) \approx 1 - j \frac{2}{\pi} \left( \ln \left( \frac{x}{2} \right) + \gamma \right) \quad \leftarrow x \ll 1 \rightarrow \quad \Uparrow \quad H_1^{(2)}(x) \approx \frac{x}{2} + \frac{2j}{\pi} \frac{1}{x} \]

\[ \lim_{\varepsilon \to 0} \int_{a}^{\frac{1}{\varepsilon}} \frac{1}{x} \, dx = \ln x \bigg|_{\varepsilon}^{a} = \ln a - \ln \varepsilon \]

\[ = a \ln a - a \quad \text{convergent} \]

Thus the singularity of \( g \) is integrable but not of \( \nabla g \).
What happens when you integrate past a singularity?

**Improper integral** e.g. \( \int_{-a}^{b} \frac{1}{x} \, dx \) and both \( a, b > 0 \). Since \( \frac{1}{x} \to \infty \) as \( x \to 0 \),

Rewrite as: \( \int_{-a}^{b} \frac{1}{x} \, dx = \int_{-a}^{-\varepsilon} \frac{1}{x} \, dx + \int_{-\varepsilon}^{b} \frac{1}{x} \, dx \)

If BOTH \( \eta, \varepsilon \) approach zero independently, and the limit exists, then we say the integral is convergent. Is that true here?

\[ \int_{-\varepsilon}^{b} \frac{1}{x} \, dx + \int_{-\varepsilon}^{b} \frac{1}{x} \, dx = (\ln \eta - \ln a) + (\ln b - \ln \varepsilon) \to \int_{-a}^{-\varepsilon} \frac{1}{x} \, dx + \int_{-\varepsilon}^{b} \frac{1}{x} \, dx \]

\[ = \ln \frac{b}{a} \]

No! It is divergent. But this exists →

Called the Cauchy principle value (PV) of the integral

\( \text{PV} \int_{-\varepsilon}^{b} f(x) \, dx \)
Back to the surface integral equations:

$$\oint [g_1(r, r') \nabla \phi(r) \cdot \hat{n} - \phi(r) \nabla g_1(r, r') \cdot \hat{n}] \, dl = \phi_i(r'), \quad r' \in V_2$$

$$\oint [g_2(r, r') \nabla \phi(r) \cdot \hat{n} - \phi(r) \nabla g_2(r, r') \cdot \hat{n}] \, dl = 0, \quad r' \in V_1$$

How do we change the integration contours?
Putting it together: evaluating the integrals

\[ \int_0^l dl = l \]

\[ \int \mathbf{E} \cdot d\mathbf{l} = \pi \varepsilon \]

\[ \int_0^\pi \varepsilon d\theta = -\pi \varepsilon \times \]

\[ \int_0^\pi \varepsilon (-d\theta) = \pi \varepsilon \]

\[ I_1 = \lim_{\varepsilon \to 0} -\frac{j k \varepsilon}{4} \times \frac{j^2}{\pi k} \varepsilon = -\frac{1}{2} \]

\[ I_2 = \int_0^l \mathbf{E} \cdot d\mathbf{l} = \frac{j k}{4} \int_0^\pi H_1^{(2)}(k \rho) (-1) (-\varepsilon d\theta) \]

\[ = +\frac{1}{2} \]
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Recap what we already know to solve:

\[ \nabla^2 \phi(r) + k^2 \phi(r) = f(r) = j \omega \mu J(r) \quad \text{(1)} \]
\[ \nabla^2 g(r, r') + k^2 g(r, r') = -\delta(r, r') \quad \text{(2)} \]

\[ L = \nabla^2 + k^2 \]

\[ \alpha = 0 \]

To solve this:

\[ x^2 y'' + xy' + (x^2 - x^2)y = 0 \quad \text{(Bessel's Diff Eqn)} \]

\[ f(r) = -\int g(r, r') f(r') \, dr' + \text{const.} \quad \text{Homogeneous object} \]
Volume Integral Equations: Setting up

How is our current problem different?

When there is no object:
\[ \nabla^2 E_i(r) + k_0^2 E_i = j \omega \mu J_z(r) \quad (1) \]

Add the object \( V_2 \)
\[ \nabla^2 E(r) + k_0^2 \epsilon_r(r) E(r) = j \omega \mu J_z(r) \quad (2) \]

\[ \nabla^2 [E(r) - E_i(r)] + k_0^2 \epsilon_r(r) E(r) - k_0^2 E_i(r) = 0 \quad (3) \]

\[ \nabla^2 g(r, r') + k_0^2 g(r, r') = -\delta(r, r') \quad (4) \]
\[ \nabla^2 \phi + k_0^2 \phi = f(r) \]
Volume Integral Equations: Solving

Get it into a form that we can solve:

\[ \nabla^2 (E - E_i) + k_0^2 \varepsilon_r E - k_0^2 E_i + k_0^2 E = k_0^2 E \]

Using

\[ \nabla^2 g(r, r') + k_0^2 g(r, r') = -\delta(r, r') \]

we get

\[ E(r) - E_i(r) = \int g(r, r') k_0^2 (\varepsilon_r(r') - 1) E(r') \, dr' \]

known: \( E_i(r), \varepsilon_r(r), g(r, r') \)

unknown: \( E(r) \)

Fredholm integral eqn of 2nd kind

\[ E(r) - \int g(r, r') k_0^2 (\varepsilon_r(r') - 1) E(r') \, dr' = E_i(r) \]

\[ \uparrow \]

Find \( E(r) \) inside \( V_2 \) \rightarrow choose \( r \in V_2 \)

2 steps:

1) Find \( E(r) \) inside \( V_2 \) \rightarrow choose \( r \in V_2 \)

2) Find \( E(r) \) anywhere \rightarrow choose \( r \in V_1 \)
Volume Integral Equations: Solving (MoM)

Use MoM: Pulse basis, delta testing

\[ t_m(r) = \delta(r-r_m) \quad (2D \ delta \ function) \]

\[ E(r) = \sum_{n=1}^{N} a_n P_n(r) \]

To solve for:

\[ E(r) = \int k_0^2 g(r, r') \chi(r')E(r')dr' = E_i(r) \]

1) Pulses:

\[ \sum_{n=1}^{N} a_n P_n(r) - \int k_0^2 g(r, r') \sum_{n=1}^{N} x_n a_n P_n(r) dr' = E_i(r) \]

2) Testing:

\[ \int (\cdot) \delta(r-r_m) dr \Rightarrow a_m - \int k_0^2 g(r_m, r') \sum_{n=1}^{N} x_n a_n P_n(r') dr' = E_i(r_m) \]

\[ -j \frac{H_0}{4} (k |r-r_m|) \]

\[ n^{th} \ pulse \]
Volume Integral Equations: Solving (contd)

Any problems with singularities here?

2 cases

\[ \gamma_m \in n^{th} \text{ pulse } (m \neq n) \]
\[ \gamma_m \in n^{th} \text{ pulse } (m = n) \rightarrow \text{ potential singularity} \]

\[ \int g(\gamma_m, y') \, dy' \]

\[ \int \int H_o^{(2)}(k \rho) \, d\rho \, d\theta = \begin{cases} 
\frac{2\pi a}{k} \int J_1(k\rho) H_0^{(2)}(k \rho_{mn}) & m \neq n \\
\frac{2}{k^2} \left[ \pi k a H_1^{(2)}(ka) - 2j \right] & m = n.
\end{cases} \]

\[ \rho = \gamma_m + \delta \]
Volume Integral Equations: Summary

Putting it all together:

\[ a_m = \sum_{n=1}^{N} a_n x_n k_0^2 \int g(r_m, r') p_n(r') \, dr' = E_i(r_m) \]

System:

\[
\begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} b \\ \vdots \\ b \end{bmatrix}
\]

\[
\begin{bmatrix} E_i(r_1) \\ \vdots \\ E_i(r_N) \end{bmatrix}
\]

\[ E(r) = E_i(r) + k_0^2 \int g(r, r') x(r') E(r') \, dr' \]

\[ v < v_i \]

Known

Total = Incident + Scattered.

Subs. \[ E(r) = \sum_{n=1}^{N} a_n p_n(r') \]

Known
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References:

- Ch 8 of ‘Waves and fields in inhomogeneous media’, Chew
- Ch 2.5 of ‘Computational Methods for Electromagnetics’, Peterson, Ray, Mitra