Computational Electromagnetics : Introduction to Green's functions

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Topics in this module

1. Motivations for Green's functions
2. A one-dimensional example
3. Some general properties of Green's functions
4. A two-dimensional example
5. A three-dimensional example

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Green's function: the motivation

Electrical Engineers are familiar with the concept of a impulse response of a system:

LTI

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Fourier transform defn:
$X(\omega) = \int_{-\infty}^{\infty} x(t) \exp(j\omega t)\, dt$

If $X(\omega) = 1$, $Y(\omega) = H(\omega)$

$\updownarrow$

$x(t)$? $\delta(t)$

Domain: time freq

$$y(t) = h(t) * x(t) \quad \longleftrightarrow \quad Y(\omega) = H(\omega)X(\omega)$$

How do we calculate $h(t)$?

1) Calc $\left(\frac{Y(\omega)}{X(\omega)}\right)$

2) IFT $H(\omega) \rightarrow h(t)$

Green's function: the motivation

Make the idea of impulse response more general → also called **Green's** function

resp
$\downarrow$ i/p → given

Now $\mathbb{L}$ is an operator: $\mathbb{L}\,\phi(r) = f(r)$ — ①

unknown

$\mathbb{L}$ acts on $\boxed{\text{unprimed}}$ only. Compare ③ & ①

$$\boxed{\phi(r) = \mathbb{L}^{-1} f(r)}$$

$\rightarrow \nabla^2,\ \nabla^2 + k^2,\ \frac{\partial}{\partial z},\ r' \rightarrow \frac{\partial}{\partial r}$

Define impulse response as: $\mathbb{L}\, g(r, r') = \delta(r, r')$ ← ②

$$\Rightarrow \boxed{\phi(r) = \int f(r')\, g(r, r')\, dr'}$$

Given g, what is ϕ?

$f(r') \times$ ②

$f(r')\,\mathbb{L}\, g(r, r') = f(r')\,\delta(r, r')$

How to solve:

$\Rightarrow \mathbb{L}\, f(r') g(r, r') = f(r')\,\delta(r, r')$

Integrate over a region incl $r = r'$ over primed coordinates

$\mathbb{L} \underbrace{\int f(r') g(r, r') dr'}{} = f(r)$ — ③

(this is the equivalent of convolution)

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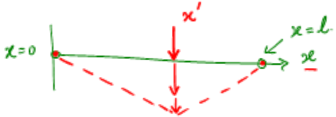
1-D example: string tied at both ends



Differential equation is $\boxed{\dfrac{d^2u(x)}{dx^2} = F(x)}$

$u(x)$: String displacement at x.

$F(x)$: Applied force.

$L \equiv \dfrac{d^2}{dx^2}$

$\phi \equiv u$

$f \equiv F$

Boundary conditions are: At both ends $u(0) = 0$

$u(\ell) = 0$

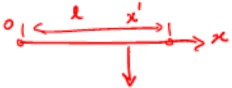
Green's function defn:

$$\boxed{\frac{d^2 G(x,x')}{dx^2} = \delta(x,x') = \delta(x-x')}$$

s.t. $G(x=0,x') = G(0,x') = 0$

$G(x=\ell, x') = G(\ell, x') = 0$

$x \neq x'$.



1-D example: solving with boundary conditions

Let's solve when $x \neq x'$ $\implies$ $\frac{d^2 g(x,x')}{dx^2} = 0$ $\implies$ $Ax + B = G(x,x')$

Consider two cases:

1) $x < x'$

$G'' = 0$

2) $x > x'$

$$G(x,x') = \begin{cases} A_1 x + B_1 & — (1) \\ A_2 x + B_2 & — (2) \end{cases}$$

How many variables?

4

→ 3 conds so far.

$A_2 x + B_2 = A_2 (x - l)$

Apply boundary condns

$x = 0$: $A_1 \times 0 + B_1 = 0 \Rightarrow B_1 = 0$ — (a

$x = l$: $A_2 l + B_2 = 0 \Rightarrow B_2 = -A_2 l$. — (b

String continuity

$x = x' \Rightarrow A_1 x' = A_2 (x' - l)$

Case 1

$\Rightarrow A_2 = \frac{A_1 x'}{x' - l}$ — (c

1-D example: final solution

$G'' = \delta(r, r')$

We have 4 variables, and 3 relations. Final trick?

Integrate. $\int_{x'-\varepsilon}^{x'+\varepsilon} G''(x,x')\,dx = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\,dx$

$G'(x,x')\Big|_{x'-\varepsilon}^{x'+\varepsilon} = 1 = A_2 - A_1$ — (d

$G' = \begin{cases} A_1 & x < x' \\ A_2 & x > x' \end{cases}$

$A_1 = \dfrac{x'-l}{l}, \quad A_2 = \dfrac{x'}{l}$

Is G' continuous?

Final solution is:

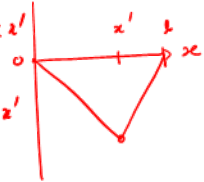
$u(x) = \int_0^l G(x,x')F(x')\,dx'$

F

Wrapping it up:

A_1

$$G(x,x') = \begin{cases} \dfrac{(x'-l)}{l}\,x &, x < x' \\ \left(\dfrac{x-l}{l}\right) x' &, x > x' \end{cases}$$



1-D example: alternate representation

We derived a closed form solution, but alternatives possible

$G(x,x')$ has finite energy $\implies$ square integrable

Write as: $G(x,x') = \sum_{n=1}^{\infty} a_n(x') \sin\left(\frac{n\pi x}{l}\right)$

$$\sum_{n=1}^{\infty} -\frac{n^2\pi^2}{l^2} a_n(x') \sin\left(\frac{n\pi x}{l}\right) = \delta(x,x')$$

Substitute into eqn: $G''(x,x') = \delta(x,x')$

How to get a_n? Orthogonality? $\int_0^l \sin\left(\frac{m\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) dx = \begin{cases} l/2, & m=n \\ 0, & m \neq n \end{cases}$

$$-\frac{m^2\pi^2}{l^{\cancel{2}}} a_m(x') \times \frac{\cancel{l}}{2} = \sin\left(\frac{m\pi x'}{l}\right)$$

Finally we get $\boxed{G(x,x') = -\frac{2l}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(\frac{n\pi x'}{l}) \sin(\frac{n\pi x}{l})}$

Series

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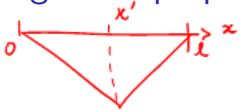
Green's functions: general properties

Keep as template: $G(x,x') = \begin{cases} \frac{(x'-l)}{l}x & x < x' \\ \frac{(x-l)}{l}x' & x > x' \end{cases}$



Following properties are true of Green's functions in general:

1) Homogeneous diff^n eqn → Satisfies it. ✓
2) Symmetric w.r.t. x, x' ✓
3) Satisfies Homogeneous boundary conds ✓
4) It is continuous at $x = x'$
5) G' has a discontinuity at $x = x'$.

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2-D example: the wave equation

Already seen this wave equation:

$$\boxed{\nabla^2\phi(r) + k^2\phi(r) = f(r)} \quad ①$$

$$\left[\ \phi(r) = -\int_{V'} f(r')\, G(r,r')\, dr'\ \right]$$

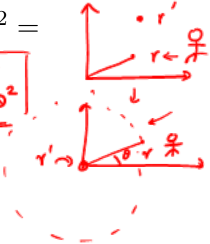
And the corresponding Green's fn defn:

$$\boxed{\nabla^2 G(r,r') + k^2 G(r,r') = -\delta(r,r')} \quad ②$$

$\times f(r')$

$\xleftarrow{\text{Intg}}$ $\nabla^2 f(r')\, G(r,r') + k^2 G(r,r') f(r') = -f(r')\,\delta(r,r')$

In 2D polar coordinates: $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \boxed{\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}}$



To solve, start with $r' = 0$ and consider $r > 0$

w.l.o.g. we have angular symmetry.

$\Rightarrow$ No θ depn.

$$\rightarrow \frac{\partial^2}{\partial r^2} G(r) + \frac{1}{r}\frac{\partial}{\partial r} G(r) + k^2 G(r) = 0$$

implicit $\rightarrow G(r,0)$.

2-D example: polar coordinates soln

Mult by r^2

Our eqn: $\underbrace{r^2 \frac{d^2}{dr^2}G(r) + r\frac{d}{dr}G(r) + k^2r^2G(r) = 0}_{r}$

Bessel's eqn: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-\alpha^2)y = 0$

const.

$\alpha = 0$, $kr = x$

$\frac{d\psi}{dr} = \frac{d\psi}{dx}\frac{dx}{dr} = k\frac{d\psi}{dx}$

$\frac{x^2}{k^2}k^2\frac{d^2}{dx^2}G(\frac{x}{k}) + \frac{x}{k}\cdot k\cdot\frac{d}{dx}G(\frac{x}{k}) + x^2G(\frac{x}{k}) = 0$

$x^2\frac{d^2}{dx^2}G(\frac{x}{k}) + x\frac{d}{dx}G(\frac{x}{k}) + x^2G(\frac{x}{k}) = 0$

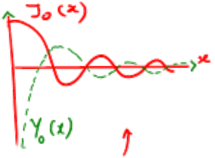
First kind Second kind

1) Solns are: $J_\alpha(x)$ $Y_\alpha(x)$

Hankel fn

2) Also: $H_\alpha^{(1)}(x)$ $H_\alpha^{(2)}(x)$

$J + jY$ $J - jY$



General soln:

$\boxed{G(\frac{x}{k})} = a H_0^{(1)}(x) + b H_0^{(2)}(x)$

$\boxed{G(r) = a H_0^{(1)}(kr) + b H_0^{(2)}(kr)}$

J_0, Y_0

2-D example: boundary conditions

Which form of the solution to take, and why? What have we not considered so far?

$$G(r) = aH_0^{(1)}(kr) + bH_0^{(2)}(kr)$$ general.

But at large r?

$$H_0^{(1)}(x) \approx \sqrt{\frac{2}{\pi x}} \exp(j(x - \tfrac{\pi}{4}))\, e^{j\omega t}$$

incoming $j(x+\omega t)$

$$H_0^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}} \exp(-j(x - \tfrac{\pi}{4}))\, e^{j\omega t}$$

outgoing $-j(x-\omega t)$

The observer only sees outgoing wave

$\Rightarrow a = 0$

Finally, $G(r) = b\, H_0^{(2)}(kr)$

2-D example: evaluating constants

How do we evaluate b?

Recall: $\underbrace{\iint_{S_\varepsilon} \nabla^2 G(r)}_{\to (a)} + \underbrace{k^2 G(r)\, ds}_{\to (b)} = \underbrace{\iint_{S_\varepsilon} -\delta(r)\, ds}_{(c)}$

$\lim \varepsilon \to 0$

$\varepsilon << 1$

$r\,dr\,d\theta$

Term (a): $\iint \nabla^2 G(r)\, ds = 2\pi \int \nabla^2 G(r)\, r\,dr$

$H_0^{(2)}(x) \to \quad J_0(x) \simeq 1 \quad , x << 1$

$Y_0(x) \simeq \frac{2}{\pi} \ln x \quad , x << 1$

$\nabla^2 G = \nabla \cdot \nabla G$

$\nabla G = \frac{\partial G}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial G}{\partial \theta}\hat{\theta}$

$\iint \nabla \cdot \nabla G\, ds = \oint \nabla G \cdot \hat{n}\, dl = \oint \left(\frac{\partial G}{\partial r}\right) dl$

$= \oint \frac{\partial}{\partial r}\left[1 - j\frac{2}{\pi} \ln kr\right] b\, dl = \oint -\frac{2jk}{\pi} \times \frac{1}{kr}\, dl = -\frac{2jk}{\pi} \times \frac{1}{kr} \times 2\pi r \Big|_{r=\varepsilon} b$

$= -4jb \leftarrow (a)$

Term (b):

$k^2 \iint G(r)\, r\,dr\,d\theta = 2\pi k^2 \int G(r)\, r\,dr = 2\pi k^2 \int r\left(1 - j\frac{2}{\pi}\ln kr\right) b\, dr$

$= 2\pi k^2 b \left[\underbrace{\int_0^\varepsilon r\,dr}_{\to 0 \text{ as } \varepsilon \to 0} - j\frac{2}{\pi}\left[\int_0^\varepsilon r \ln kr\, dr\right]\right]$

$= \ln kr \cdot \frac{r^2}{2}\Big|_0^\varepsilon - \underbrace{\int_0^\varepsilon \frac{1}{r} \cdot \frac{r^2}{2}\, dr}_{\to 0}$

$r^2 \ln kr$

$\lim_{r \to 0} \frac{\ln kr}{1/r^2} = \frac{1/r}{-2/r^3} = \frac{-r^2}{2}$

$r = 0 \longrightarrow 0$

$\varepsilon \longrightarrow -\frac{\varepsilon^2}{2}$

term(b) = 0

2-D example: evaluating constants

How do we evaluate b?

Recall: $\int_{S_\epsilon}[\nabla^2 G(r) + k^2 G(r)]\,dS = \int_{S_\epsilon} -\delta(r)\,dS$

$\iint \delta(x,y)\,dx\,dy = 1$

-1

$\cancel{\iint \delta(r)\, r\,dr\,d\theta = 2\pi \times \int r\,\delta(r)\,dr}$

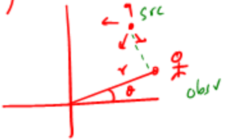
Term (c): $= -1$

$-4jb + 0 = -1$

$b = \frac{1}{4j} = \frac{-j}{4}$

$G(r) = \frac{-j}{4} H_0^{(2)}(kr)$

$G(\vec{r}) = \frac{-j}{4} H_0^{(2)}(k|\vec{r}|)$



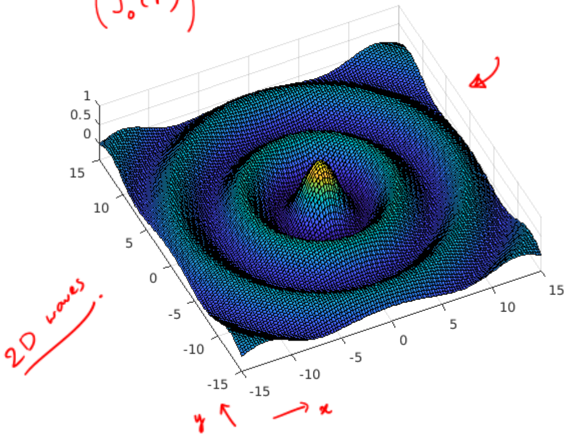
Putting it all together:
$G(r) =$

Finally, $G(r, r') =$ $\boxed{\frac{-j}{4} H_0^{(2)}(k|\vec{r} - \vec{r}'|)}$

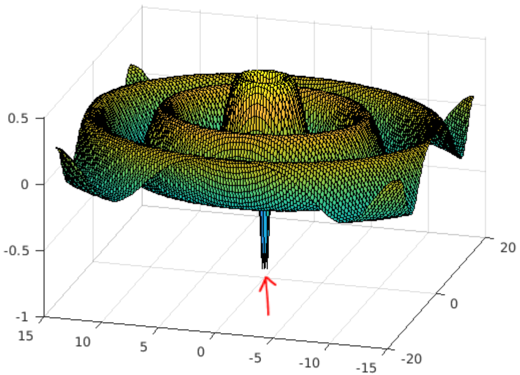
$e^{j(kx-\omega t)}$

2-D example: visualizing the wave

$(J_0(\bar{r}))$



$Y_0(\bar{r})$




```
[X,Y] = meshgrid(-15:0.25:15,-15:0.25:15);
R = sqrt(X.^2+Y.^2); BJ = besselj(0,R);
surf(X,Y,BJ)
```

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3-D example: the wave equation

Same (wave) equation: $\underbrace{\nabla^2 G(r) + k^2 G(r) = -\delta(r)}$ Set $\underbrace{r' = 0}$

In spherical polar coordinates, $r-$depn terms are: $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)$

Simplifying for $r > 0$: $\nabla^2 G + k^2 G = 0$

$e^{j\omega t}$

$\Rightarrow a = 0 \rightarrow G(r) = \frac{b\,e^{-jkr}}{r}$

Solving: Boundary conditions?

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dG}{dr}\right) + k^2 G = 0$$

$$\frac{1}{r}\frac{d}{dr}\left(r^2 \frac{dG}{dr}\right) + k^2 r G = 0 = 2\frac{dG}{dr} + r\frac{d^2G}{dr^2} + k^2 r G = 0$$

Final form: $\frac{d^2}{dr^2}(rG) + k^2 r G = 0$

$$r\,G(r) = a\,e^{jkr} + b\,e^{-jkr}$$

$$G(r) = a\frac{e^{jkr}}{r} + b\frac{e^{-jkr}}{r}$$

spherical plane waves.

$G = b\,\frac{e^{-jkr}}{r}$

3-D example: evaluating the constant

$\hat n = \hat r$

Integrate both sides: $\int_V \left(\nabla^2 G(r) + k^2 G(r)\right) dv = \int_V \underbrace{-\delta(r)\, dv}_{=-1}$

First term:

$\nabla^2 G = \nabla \cdot \nabla G$

$\int_V \nabla \cdot \nabla G\, dv = \oint \nabla G \cdot \hat n\, ds$

$\nabla G = \frac{\partial G}{\partial r}\hat r$

$= b \oint \left[\frac{-jke^{-jkr}}{r} - \frac{e^{-jkr}}{r^2} \right] r^2 \sin\theta\, d\theta\, d\phi$

$= -4\pi b \left[jk\frac{e^{-jkr}}{r} + \frac{e^{jkr}}{r^2} \right]_{r=\varepsilon} r^2$

$\varepsilon \to 0$

$= -4\pi b$

Second term: $\int_0^\varepsilon k^2 b \frac{e^{-jkr}}{r} r^2 4\pi\, dr = 4\pi k^2 b \underbrace{\int_0^\varepsilon r e^{-jkr}\, dr}$

as $\varepsilon \to 0 \quad = 0$

$-4\pi b = -1, \quad b = \frac{1}{4\pi}$

$R = |\bar r - \bar r'|$

Final expression: $G(r) = \frac{1}{4\pi r} e^{-jkr}$

$$G(\bar r, \bar r') = \frac{1}{4\pi} \frac{e^{-jk|\bar r - \bar r'|}}{|\bar r - \bar r'|} \Bigg\} 3D$$

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Reference: Ch 14 of Advanced Engineering Electromagnetics, Balanis