Computational Electromagnetics : Surface Integral Equations

Uday Khankhoje

Electrical Engineering, IIT Madras

Topics in this module

1. Motivation: Bending Waves
2. Setting up the Helmholtz Equation
3. Solving the Helmholtz Equation: Green's functions
4. Huygen's principle & the Extinction theorem
5. Formulating the integral equations

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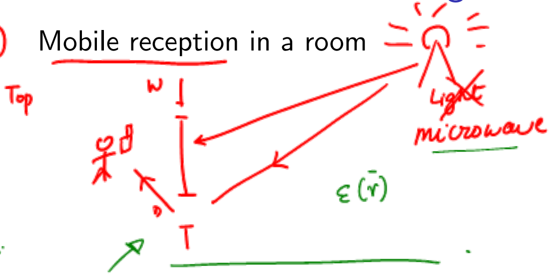
Observations of waves bending

1 Mobile reception in a room



2 Slit-light interference

Huygen's principle in school

Interference pattern.

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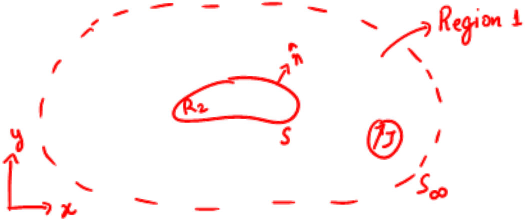
Problem setup

A two region problem: a source and an object



Maxwell's equations (any volume):

✓ $\nabla \times \vec{E}(\vec{r}) = -j\omega\mu_0 \vec{H}(\vec{r})$ → $\nabla\times(\nabla\times\bar{E}) = \nabla(\nabla\cdot\bar{E}) - \nabla^2\bar{E} = -\nabla^2\bar{E}$

✓ $\nabla \times \vec{H}(\vec{r}) = j\omega\epsilon(\vec{r})\,\vec{E}(\vec{r}) + \vec{J}(\vec{r})$

$-\nabla^2\bar{E} = -j\omega\mu_0\left(j\omega\epsilon(r)\bar{E} + \bar{J}\right)$

$$\Longrightarrow \quad \nabla^2\vec{E}(\vec{r}) + \boxed{\omega^2\mu_0\epsilon(\vec{r})}\vec{E}(\vec{r}) = j\omega\mu_0\vec{J}(\vec{r})$$ ✓

Helmholtz equation

True for Regs 1, 2.

Polarizations

choose

1) TM → (E_z, H_x, H_y) → $H_z = 0$
2) TE → (H_z, E_x, E_y) → $E_z = 0$

$\nabla\cdot\bar{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$
$= 0 + 0 + 0$

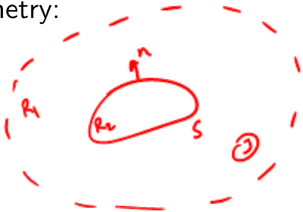
Assumptions

Non magnetic media

2D problem → $\frac{\partial}{\partial z} = 0$, fns (x,y)

Helmholtz equation: Making it concrete

Geometry:



Equations:

Reg 1: $\nabla^2 \phi_1(r) + k_1^2 \phi_1(r) = j\omega\mu_0 J_z(r)$

Reg 2: $\nabla^2 \phi_2(r) + k_2^2 \phi_2(r) = 0$

$$\omega^2 \mu_0 \underline{\epsilon}(r) = \underline{k}^2$$

Material properties:

$$\epsilon(r) \to \epsilon(x, y)$$

Fields, currents: TM pol: $\overbrace{E_z, H_x, H_y}^{\text{scalar fns}}$

$E_z \to$ Reg 1 $\to \phi_1(r)$

$\hookrightarrow$ Reg 2 $\to \phi_2(r)$

$$\vec{J} = J_z \hat{z}$$

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Introduce a new "Green's" function

Want to solve:

$\vec{r}$

$$\nabla^2\phi_1(r) + k_1^2\,\phi_1(r) = Q(r)$$
$$\nabla^2\phi_2(r) + k_2^2\,\phi_2(r) = 0$$

$\nabla^2(\,) + k^2(\,) = (\,)$

Introduce two new functions $\boxed{g_1,\ g_2}$ s.t.:

$$\nabla^2 g_1(r, r') + k_1^2\, g_1(r, r') = -\delta(r - r')$$
$$\nabla^2 g_2(r, r') + k_2^2\, g_2(r, r') = -\delta(r - r')$$

only r

dirac delta fn

[assume we know these (for now)]

Some algebra: $[(\ ) \times g_1 - (\ ) \times \phi_1]$:

← integrate over V_1 : volume of Region 1

$$\int_{V_1} dV \left(g_1 \nabla^2\phi_1 - \phi_1 \nabla^2 g_1 = g_1 Q + \phi_1 \delta(r - r')\right)$$

$$\cancel{k_1^2\phi_1 g_1} - \cancel{k_1^2 g_1\phi_1}$$

$$\int_{V_1} g_1(r, r')\, Q(r)\, dV = -\phi_1(r')$$

Vector calculus results;

$$\nabla \cdot (g\nabla\phi) = \nabla g \cdot \nabla\phi + g\nabla^2\phi$$

$$\Longrightarrow \quad g_1\nabla^2\phi_1 - \phi_1\nabla^2 g_1 = \nabla\cdot(g_1\nabla\phi_1 - \phi_1\nabla g_1)$$

. . . some more vector calculus

Recall divergence theorem, apply to region 1:

$\int \nabla\cdot\vec{A}'\,dv = \oint \vec{A}'\cdot d\vec{s}$ (3D)

$\vec{A}'\cdot\hat{n}\,ds$

$\int_S \nabla\cdot\vec{f}\,dS = \oint \vec{f}\cdot\hat{n}\,dl$ (in 2D)

BC

$\int \nabla\cdot\vec{f}\,ds = \oint_S \vec{f}\cdot\hat{n}_1\,dl + \oint \vec{f}\cdot\hat{n}_\infty\,dl$ $\searrow 0$

$= -\oint_S \vec{f}\cdot\hat{n}\,dl$

$$\Longrightarrow \int_{V_1}[g\nabla^2\phi - \phi\nabla^2 g]\,dS = \int \nabla\cdot[g\nabla\phi - \phi\nabla g]\,dS = -\oint_S (g\nabla\phi - \phi\nabla g)\cdot\hat{n}\,dl$$

Putting it together:

$\to \int_{V_1} \phi_1(r)\,\delta(r-r')\,dv$

$$\phi_i(r') - \oint[g_1(r, r'\nabla\phi_1(r) - \phi_1(r)\nabla g_1(r,r')]\cdot\hat{n}\,dl$$

$$= \begin{cases} \phi_1(r'), & r' \in V_1 \\ 0, & r' \in V_2 \end{cases}$$

$r' \in V_1$ → Huygen's principle

$r' \in V_2$ → Extinction theorem.

Reviewing the derivation so far

$$\int_{V_1} \left(g_1 \nabla^2 \phi_1 - \phi_1 \nabla^2 g_1\right) ds = \int \left(g_1(r,r')\, Q(r) + \phi_1(r)\,\delta(r-r')\right) ds$$

– 2D problem
→ 3D $\int dv$

$$\int_{V_1} \nabla\cdot\left(g_1 \nabla\phi_1 - \phi_1 \nabla g_1\right) ds = -\phi_i(r') + \int_{V_1} \underbrace{\phi_1(r)\,\delta(r-r')\, ds}$$

$$-\oint_S \left(g_1 \nabla\phi_1 - \phi_1 \nabla g_1\right)\cdot \hat{n}\, dl$$

inward normal

$$= \begin{cases} \phi_1(r') & \text{if } r' \in V_1 \\ 0 & \text{if } r' \in V_2 \end{cases}$$

Huygen's principle.

Physical Interpretations:

$r' \in V_1$
$r' \in V_2$

Total

$$\begin{Bmatrix} \phi_1(r') \\ 0 \end{Bmatrix} = \underbrace{\phi_i(r')}_{\text{incident field, no object}} - \underbrace{\oint_S \left(g_1 \nabla\phi_1 - \phi_1 \nabla g_1\right)\cdot \hat{n}\, dl}_{\text{scattered field due to object}}$$

Extinction thm.

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Huygen's principle & the Extinction theorem

ϕ_1

$$\phi_i(r') - \oint [g_1(r,r')\nabla\phi_1(r) - \phi_1(r)\nabla g_1(r,r')] \cdot \hat{n}\, dl$$

$$= \begin{cases} \phi_1(r') & r' \in V_1 \rightarrow \text{Huy} \\ 0 & r' \in V_2 \rightarrow \text{Ext} \end{cases}$$

Similarly for region 2:

ϕ_2

$$\oint [g_2(r,r')\nabla\phi_2(r) - \phi_2(r)\nabla g_2(r,r')] \cdot \hat{n}\, dl$$

boundary

$$= \begin{cases} \phi_2(r') & r' \in V_2 \rightarrow \text{Huy} \rightarrow \text{inside } V_2 \\ 0 & r' \in V_1 \rightarrow \text{Ext} \end{cases}$$

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Formulating the integral equations – 1

If we pick the bottom equations of each pair from before:

$$\phi_i(r') = \oint_S [g_1(r,r')\nabla\phi_1(r) - \phi_1(r)\nabla g_1(r,r')]\cdot\hat{n}\,dl$$

$$0 = \oint_S [g_2(r,r')\nabla\phi_2(r) - \phi_2(r)\nabla g_2(r,r')]\cdot\hat{n}\,dl$$

How many variables?

$\underbrace{\phi_1, \phi_2}_{\downarrow\ \phi}, \nabla\phi_1\cdot\hat{n}, \nabla\phi_2\cdot\hat{n}$

4

Boundary conditions? $\vec{E}_{tan_1} = \vec{E}_{tan_2}$, $\vec{H}_{tan_1} = \vec{H}_{tan_2}$.

TM: $\vec{E} = E_z\hat{z} = \phi\hat{z}$

$\phi_1(r') = \phi_2(r')$ on S.

$= \phi(r')$ on S.

$\nabla\phi\cdot\hat{n} = \left(n_x\frac{\partial\phi}{\partial x} + n_y\frac{\partial\phi}{\partial y}\right)$ $\quad \hat{n} = (n_x, n_y, 0) \rightarrow \hat{t} = (-n_y, n_x, 0)$ $\quad \Rightarrow \underline{\nabla\phi_1\cdot\hat{n} = \nabla\phi_2\cdot\hat{n} \text{ on } S.}$

Formulating the integral equations – 2

If we pick the bottom equations of each pair from before:

$$\phi_i(r') = \oint [g_1(r,r')\nabla\phi_1(r) - \phi_1(r)\nabla g_1(r,r')]\cdot\hat{n}\,dl$$

$$0 = \oint [g_2(r,r')\nabla\phi_2(r) - \phi_2(r)\nabla g_2(r,r')]\cdot\hat{n}\,dl$$

How many variables?

$\left(n_y^2\frac{\partial\phi}{\partial y} + n_x n_y\frac{\partial\phi}{\partial x},\ -n_x^2\frac{\partial\phi}{\partial x} - n_x n_y\frac{\partial\phi}{\partial y},\ 0\right)$

$= (n_y\,\underline{\nabla\phi\cdot\hat{n}},\ -n_x\,\underline{\nabla\phi\cdot\hat{n}},\ 0)$

$= (\nabla\phi\cdot\hat{n})(n_y, -n_x, 0) = (\nabla\phi\cdot\hat{n})\hat{t}$

$\underline{\vec{H}_{tan} = \frac{j}{\omega\mu_0}(\nabla\phi\cdot\hat{n})\hat{t}.}$

Boundary conditions? $\quad \vec{H}_{tan_1} = \vec{H}_{tan_2}.$

$\nabla\times\vec{E} = -j\omega\mu_0\vec{H}$

$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & \phi(r) \end{vmatrix}$, $\quad \vec{H} = \frac{j}{\omega\mu_0}\left(\frac{\partial\phi}{\partial y}, -\frac{\partial\phi}{\partial x}, 0\right)$

$\phi(r) \rightarrow (x,y)$

$= \left(\frac{\partial\phi}{\partial y}, -\frac{\partial\phi}{\partial x}, 0\right)$

$= H_t\hat{t} + H_n\hat{n}$

$\vec{H}_{tan} = H_t\hat{t} = \vec{H} - H_n\hat{n}$

$= \vec{H} - (\vec{H}\cdot\hat{n})\hat{n}$

$n_x^2 + n_y^2 = 1 \leftarrow \hat{n} = (n_x, n_y, 0)$

$\left[\frac{j}{\omega\mu_0}\left(n_x\frac{\partial\phi}{\partial y} - n_y\frac{\partial\phi}{\partial x}\right)\right]$

$(\vec{H}\cdot\hat{n})$

$= \frac{j}{\omega\mu_0}\left[\left(\frac{\partial\phi}{\partial y} - n_x\left(n_x\frac{\partial\phi}{\partial y} - n_y\frac{\partial\phi}{\partial x}\right)\right), \left(-\frac{\partial\phi}{\partial x} - n_y\left(n_x\frac{\partial\phi}{\partial y} - n_y\frac{\partial\phi}{\partial x}\right)\right), 0\right]$

$\underline{\nabla\phi\cdot\hat{n}}$

The final set of equations

$$\oint [g_1(r,r')(\nabla\phi(r)\cdot\hat{n}) - \phi(r)\nabla g_1(r,r')\cdot\hat{n}]\, dl = \phi_i(r')$$

$$\oint [g_2(r,r')(\nabla\phi(r)\cdot\hat{n}) - \phi(r)\nabla g_2(r,r')\cdot\hat{n}]\, dl = 0$$

2 Eqns / 2 variables.

Fredholm, 1st kind, coupled, boundary integrals

discretization

basis fns.

Generalizing the idea

Q1: What about a magnetic medium, $\mu = \mu(r)$? ✓

$$\nabla \times (\nabla \times E) = -j\omega \nabla \times (\mu(r)\bar{H}(r))$$

$$\frac{1}{\mu(r)} \nabla \times \bar{E} = -j\omega \bar{H}$$

$-\nabla^2 E$ ✓

$$\nabla \times \left(\frac{1}{\mu(r)} \nabla \times \bar{E}\right) = -j\omega (\nabla \times \bar{H})$$

$$= -j\omega \left[j\omega \epsilon(r) \bar{E} + \bar{J} \right]$$

$$\frac{1}{\epsilon(r)} \nabla \times \vec{H} = j\omega \vec{E} + \frac{\vec{J}(r)}{\epsilon(r)}$$

photonic crystals.

Maxwell's equations:

$$\nabla \times \vec{E}(\vec{r}) = -j\omega\mu(r)\vec{H}(\vec{r})$$
$$\nabla \times \vec{H}(\vec{r}) = j\omega\epsilon(\vec{r})\,\vec{E}(\vec{r}) + \vec{J}(\vec{r})$$

Q2: What about a Helmholtz like eqn in $\vec{H}(\vec{r})$? μ_0

$$\nabla \times (\nabla \times H) = \nabla \times [j\omega \epsilon(r) E + J(r)]$$

$$-\nabla^2 H$$

$\because \nabla \cdot H = 0$

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5. Formulating the integral equations

Reference: Chapter 8 of W C Chew: Waves and fields in inhomogeneous media