Computational Electromagnetics : Review of Maxwell's Equations

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Topics in this module

1. Maxwell's Equations
2. Boundary Conditions
3. Power in a field
4. Uniqueness theorem
5. Equivalence theorems

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Maxwell's equations + continuity relation

Consider real valued physical quantities: $\mathcal{E}(\vec{r},t)$, $\mathcal{H}(\vec{r},t)$, etc

$$\nabla \times \mathcal{E}(\vec{r},t) = -\frac{\partial \mathcal{B}(\vec{r},t)}{\partial t} - \mathcal{M}(\vec{r},t), \quad \text{Faraday, 1843} \qquad (1)$$

$$\nabla \times \mathcal{H}(\vec{r},t) = \frac{\partial \mathcal{D}(\vec{r},t)}{\partial t} + \mathcal{J}(\vec{r},t), \quad \text{Ampere, 1823} \qquad (2)$$

$$\nabla \cdot \mathcal{D}(\vec{r},t) = \rho_e, \quad \text{Coulomb, 1785} \qquad (3)$$

$$\nabla \cdot \mathcal{B}(\vec{r},t) = \rho_m, \quad \text{Gauss, 1841} \qquad (4)$$

$$\nabla \cdot \mathcal{J}(\vec{r},t) = -\frac{\partial \rho_e}{\partial t}$$

$$\nabla \cdot (\quad) = \nabla \cdot (\nabla \times H) = 0$$

ρ_m, $\mathcal{M}(\vec{r},t) \rightarrow$

- Not physical
- Mathematical convenience ✓
- Makes symmetric

Maxwell's equations: a wave example

Let's apply the equations in source/charge free vacuum

$$-\frac{\partial}{\partial t}(\nabla \times B) = -\mu_0 \frac{\partial}{\partial t}(\nabla \times H)$$

$$= -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\mathcal{E} = -\nabla^2 \mathcal{E}$$

$$\left\{\begin{array}{l} \nabla \times \mathcal{E}(\vec{r},t) = -\dfrac{\partial \mathcal{B}(\vec{r},t)}{\partial t} \\ \nabla \times \mathcal{H}(\vec{r},t) = \dfrac{\partial \mathcal{D}(\vec{r},t)}{\partial t}, \\ \nabla \cdot \mathcal{D}(\vec{r},t) = 0, \quad \mathcal{D}(\vec{r},t) = \epsilon_o \mathcal{E}(\vec{r},t) \\ \nabla \cdot \mathcal{B}(\vec{r},t) = 0, \quad \mathcal{B}(\vec{r},t) = \mu_0 \mathcal{H}(\vec{r},t) \end{array}\right.$$

constitutive

Take 1D $\frac{\partial^2 \mathcal{E}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \mathcal{E}}{\partial t^2}$ wave Eqn

$\mathcal{E} = \mathcal{E}_0 \cos(kx - \omega t + \phi_0)$ is a soln

$$k^2 \mathcal{E} = \frac{\omega^2}{c^2}\mathcal{E} \qquad \boxed{k = \omega/c}$$

Take a curl of first eqn:

$$(\nabla \times (\nabla \times \mathcal{E}(\vec{r},t))) = \left[\underbrace{\nabla(\nabla\cdot\mathcal{E})}_{0} - \nabla^2\mathcal{E}\right] \leftarrow \text{VC}$$

$$\nabla^2\left[\mathcal{E}_x \hat{x} + \mathcal{E}_y \hat{y} + \mathcal{E}_z \hat{z}\right]$$

Time Harmonic form

complex

Electrical Engineers prefer phasors! e.g. $\mathcal{E}(\vec{r},t) = \text{Re}[\vec{E}(\vec{r})e^{j\omega t}]$

e^{jwt}

- $\vec{E}(\vec{r}) \to$ complex
- EE / Phy conventions

e^{-jwt}

$$\nabla \times \vec{E}(\vec{r}) = -j\omega\vec{B}(\vec{r}) - \vec{M}(\vec{r}) \tag{5}$$

$$\nabla \times \vec{H}(\vec{r}) = j\omega\vec{D}(\vec{r}) + \vec{J}(\vec{r}) \tag{6}$$

$$\nabla \cdot \vec{D}(\vec{r}) = \rho_e \tag{7}$$

$$\nabla \cdot \vec{B}(\vec{r}) = \rho_m \tag{8}$$

Constitutive relations:

Ohm's law

$$\vec{D}(\vec{r}) = \epsilon(\vec{r})\vec{E}(\vec{r}), \qquad \vec{B}(\vec{r}) = \mu(\vec{r})\vec{H}(\vec{r}), \quad \vec{J}(\vec{r}) = \sigma(\vec{r})\vec{E}(\vec{r})$$

medium

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Tangential Boundary Conditions

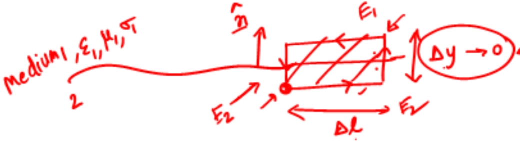
Start with $\nabla \times \vec{E}(\vec{r}) = -j\omega \vec{B}(\vec{r}) - \vec{M}(\vec{r})$

$$\iint \nabla \times \bar{E} \cdot \vec{ds} = \oint \vec{E} \cdot \vec{dl} = \iint (-j\omega \bar{B} - \bar{M}) \cdot \bar{ds}$$

$$= -(j\omega \hat{B} + \hat{M}) \Delta y \Delta l$$

$$= -\hat{M} \Delta y \Delta l$$

$$-M_s \Delta l$$



$$(E_2)_x \Delta l + (E_2)_y \frac{\Delta y}{2} + (E_1)_y \frac{\Delta y}{2} - (E_1)_x \Delta l - (E_1)_y \frac{\Delta y}{2} - (E_2)_y \frac{\Delta y}{2}$$

$$[(E_2)_x - (E_1)_x] \Delta l$$

$$\nabla \times H = j\omega D + J$$

tangential E

pure surface current

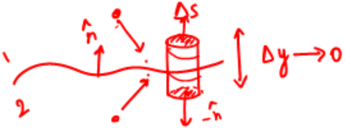
tangential H

$\hat{n} \times (\vec{E}_2 - \vec{E}_1) = -\vec{M}_s$. Similarly $\hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s$.

Normal Boundary Conditions

Start with $\nabla \cdot \vec{D}(\vec{r}) = +\rho_e$ $\longrightarrow$ $\int_v \nabla\cdot\vec{D}\,dv = \oint \vec{D}\cdot d\vec{s} = (D_1)_n \Delta s - (D_2)_n \Delta s + \cancel{(\quad)2\pi r \Delta y}$ $\to 0$

$\int \rho_e\, dv = \rho_e \Delta s(2\pi r \Delta y)$



pure surface charge.

$\hat{n}\cdot(\vec{D}_2 - \vec{D}_1) = \rho_{es}$. Similarly $\hat{n}\cdot(\vec{B}_2 - \vec{B}_1) = \rho_{ms}$.

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Power in a field

Instantaneous Poynting vector defined as $\boxed{\mathcal{S}(\vec{r},t) = \mathcal{E}(\vec{r},t) \times \mathcal{H}(\vec{r},t)}$

Use $\mathcal{E}(\vec{r},t) = Re[\vec{E}(\vec{r})e^{j\omega t}] = \frac{1}{2}[E e^{j\omega t} + E^* e^{-j\omega t}]$

$\mathcal{H} = \frac{1}{2}[H e^{j\omega t} + H^* e^{-j\omega t}]$

$e^{2j\omega t}, \quad e^{-2j\omega t}, \quad 0, \quad 0$

$S = \frac{1}{2}[Re(\bar{E} \times H^*) + Re[(E \times H) e^{2j\omega t}]]$

$\mathcal{S}_{\text{av}}(\vec{r},t) = \frac{1}{2}[Re(\vec{E}(\vec{r}) \times \vec{H}(\vec{r})^*]$

real → power

imag → reactive

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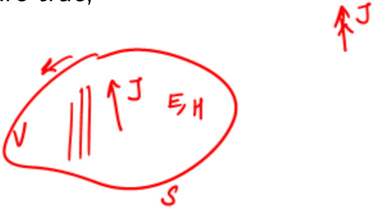
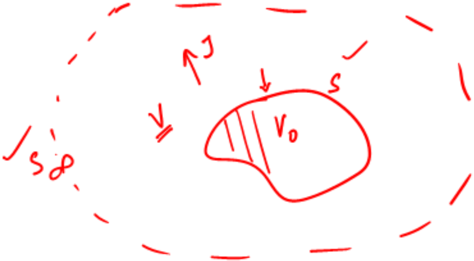
Uniqueness theorem

Statement: The field $(\vec{E}(\vec{r}), \vec{H}(\vec{r}))$ created by some sources $\vec{J}(\vec{r})$ in a lossy volume V are unique if any one of these are true;

1. $\vec{E}(\vec{r})_{tan}$ over S is known
2. $\vec{H}(\vec{r})_{tan}$ over S is known
3. $\vec{E}(\vec{r})_{tan}$ known over some part, $\vec{H}(\vec{r})_{tan}$ known over remaining part of S.





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Volume "Equivalence" Theorem

Say we have some sources in vacuum denoted by E_0, H_0

$$\text{①}\quad \begin{cases} \nabla \times \vec{E}_0(\vec{r}) = -j\omega\mu_0 \vec{H}_0(\vec{r}) - \vec{M}(\vec{r}) \\ \nabla \times \vec{H}_0(\vec{r}) = j\omega\epsilon_0 \vec{E}_0(\vec{r}) + \vec{J}(\vec{r}) \end{cases}$$

M J

$\cdots$ now place an obstacle in the medium

$$\left.\begin{aligned} \nabla \times \vec{E}(\vec{r}) &= -j\omega\mu(r)\vec{H}(\vec{r}) - \vec{M}(\vec{r}) \\ \nabla \times \vec{H}(\vec{r}) &= j\omega\epsilon(r)\vec{E}(\vec{r}) + \vec{J}(\vec{r}) \end{aligned}\right] \quad \text{②}$$

Subtract the two sets of Eqns

$$\left.\begin{aligned} \nabla \times (\vec{E}(\vec{r}) - \vec{E}_0(\vec{r})) &= -j\omega(\mu H - \mu_0 H_0) \\ \nabla \times (\vec{H}(\vec{r}) - \vec{H}_0(\vec{r})) &= +j\omega(\varepsilon E - \varepsilon_0 E_0) \end{aligned}\right\}$$

Define: $\vec{E}_s(\vec{r}) = E - E_0$ $\qquad \vec{H}_s(\vec{r})$

Scattered fields.

$E_s = E - E_0$ $\quad H = H_0 + H_s$ $\quad H_0 = H - H_s$

Volume Equivalence Theorem (contd.)

Have replaced obstacle by equivalent volume sources

$$\nabla \times (\vec{E}(\vec{r}) - \vec{E}_0(\vec{r})) = -j\omega(\mu\vec{H}(\vec{r}) - \mu_0\vec{H}_0(\vec{r})) =$$

$$= -j\omega(\mu H - \mu_0(H - H_s))$$

$$= -j\omega((\mu - \mu_0)H + \mu_0 H_s) \quad (1)$$

$$\nabla \times (\vec{H}(\vec{r}) - \vec{H}_0(\vec{r})) = +j\omega(\epsilon\vec{E}(\vec{r}) - \epsilon_0\vec{E}_0(\vec{r})) =$$

$$= +j\omega(\varepsilon E - \varepsilon_0(E - E_s))$$

$$= +j\omega((\varepsilon - \varepsilon_0)E + \varepsilon_0 E_s)$$

Define $\vec{M}_{eq}(\vec{r}) = j\omega(\mu - \mu_0)H$ $\quad \vec{J}_{eq}(\vec{r}) = j\omega(\varepsilon - \varepsilon_0)E$

Finally gives

$$\nabla \times \vec{E}_s(\vec{r}) = -\bar{M}_{eq} - j\omega\mu_0\bar{H}_s$$

$$\nabla \times \vec{H}_s(\vec{r}) = +\bar{J} + j\omega\varepsilon_0\bar{E}_s$$

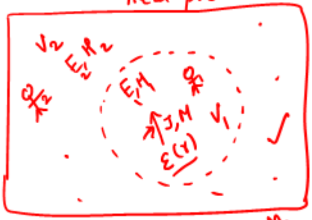
→ No object!

Price → Eqv currents

Surface Equivalence Principle

Recall tangential boundary conditions: $\hat{n} \times (\vec{E}_2 - \vec{E}_1) = -\vec{M}_s$, $\hat{n} \times (\vec{H}_2 - \vec{H}_1) = +\vec{J}_s$.

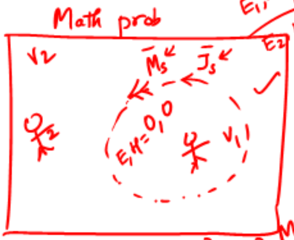
Real problem



$\hat{n} \times \bar{E}_2 = \hat{n} \times \vec{E}_1 + 0 \leftarrow M_s$

obsv 2 obsv 1

Math prob



obj replaced by surf currents.

Love's equivalence thm.

Mathematical.

$\hat{n} \times \vec{E}_2 = 0 - \vec{M}_s$ Set $= -\hat{n} \times E_1$

The discontinuity of fields is saved by the presence of surface currents

Surface Eqv: Replace object by tangential surface currents

Volume Eqv: Replace object by volume currents

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1. Maxwell's Equations
2. Boundary Conditions
3. Power in a field ✓
4. Uniqueness theorem
5. Equivalence theorems

1, 1,2,3

Reference: Chapter 7 of Advanced Engineering Electromagnetics - C A Balanis

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