Computational Electromagnetics: Review of Vector Calculus

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Topics in this module

1. Chain rule of differentiation and the gradient
2. Gradient, Divergence, and Curl operators
3. Common theorems in vector calculus
4. Corollaries of these theorems; miscellaneous results
1. Chain rule of differentiation and the gradient

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Chain rule of differentiation

- Consider a scalar function of several variables, \( f(x, y, z) \)

\[
\begin{align*}
\nabla \cdot \mathbf{r} & = \frac{\partial}{\partial x} f \mathbf{i} + \frac{\partial}{\partial y} f \mathbf{j} + \frac{\partial}{\partial z} f \mathbf{k} = \sqrt{\frac{1}{4} + \frac{1}{8} + \frac{1}{2}} \\
|\mathbf{r}| & = \sqrt{x^2 + y^2 + z^2}
\end{align*}
\]

- Want to calculate a small change in \( f \), i.e. \( df \). Say each variable has changed, e.g. \( x \rightarrow x + dx \ldots \)

- Chain rule tells us:

\[
df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz
\]

- Dot product between \( \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \) and \( (dx, dy, dz) \)
Working with the gradient

- Compact way to write change: \( df = \nabla f \cdot \vec{dl} \)

- Now we want the total change going from \( \vec{a} \) to \( \vec{b} \)

\[
\int_{\vec{a}}^{\vec{b}} df = f(\vec{b}) - f(\vec{a})
\]

- Corollary: \( \oint \nabla f \cdot \vec{dl} = 0 \)
1 Chain rule of differentiation and the gradient

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Gradient as the ‘Del’ operator

• Saw that \[ \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \]

• Generalize a ‘Del’ operator as \[ \nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \]

• Acts in three ways (like an ordinary vector)

(gradient) \( \nabla f \) (divergence) \( \nabla \cdot \vec{A} \) (curl) \( \nabla \times \vec{A} \)
Divergence: \( \nabla \cdot \vec{A} = \frac{x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \)

- Geometrically: measures how much a vector ‘diverges’ at a pt

- Examples

\[ \vec{A} = (x, y, z) \]
\[ \nabla \cdot \vec{A} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \]

\[ \vec{A} = (0, 0, 1) \]
\[ \nabla \cdot \vec{A} = 0 + 0 + \frac{\partial}{\partial z} = 0 \]

\[ \vec{A} = (0, 0, z) \]
\[ \nabla \cdot \vec{A} = 0 + 0 + \frac{\partial}{\partial z} = 1 \]
Curl: \( \nabla \times \vec{A} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \)

- Geometrically: measures how much a vector ‘swirls’ around a pt
- Examples

\[ \vec{A} = (x, y, z) \quad \vec{A} = (-y, x, 0) \quad \vec{A} = (0, x, 0) \]
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Integrals in vector calculus

• Line integrals: \( \int \vec{A} \cdot d\ell \)

• Surface integrals: \( \int \vec{A} \cdot d\vec{s} \)

• Volume integrals: \( \int f \, dv \)

\[ \int \vec{A} \, dv = \int A_x \, dv + \int A_y \, dv + \int A_z \, dv \]

\[ \rightarrow \text{ vector} \]
Divergence (a.k.a. Gauss’s / Green’s) Theorem

Geometrically:
\[
\int_V \nabla \cdot \vec{A} \, dv = \oint_S \vec{A} \cdot \vec{d}s
\]

Proof sketch:
\[
\left( \oint_{\partial V} A \cdot ds \right) = \left[ A_x(x_0 + \Delta x, y_0, z_0) - A_x(x_0 - \Delta x, y_0, z_0) \right] \Delta y \Delta z
\]

\[
\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{dV}{\Delta x, \Delta y, \Delta z} = \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}
\]

CEM : Helps reduce dimensionality of problem
Curl (a.k.a. Stoke’s) Theorem

Geometrically:
\[ \int_S (\nabla \times \vec{A}) \cdot \vec{ds} = \oint_{\Gamma} \vec{A} \cdot \vec{dl} \]

Proof sketch:
\[ \int_S (\nabla \times \vec{A}) \cdot \vec{ds} \]
Corollary:
\[ \oint_S (\nabla \times \vec{A}) \cdot \vec{ds} = 0 \]
Stoke’s Theorem in a multiply connected region

Geometrically: surface + hole

\[ \int_S (\nabla \times \vec{A}) \cdot d\vec{s} = \oint_{\Gamma_1} \vec{A} \cdot d\vec{l} - \oint_{\Gamma_2} \vec{A} \cdot d\vec{l} \]

CEM: Helps reduce domain of computation
Table of Contents

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Corollaries: Integration by parts

- Two scalar functions, \( f, g \). Know that:
  \[
  \frac{d(fg)}{dx} = f \frac{dg}{dx} + g \frac{df}{dx}
  \]
  Rearranging, integrating:
  \[
  \int_a^b f \frac{dg}{dx} \, dx = \int_a^b \frac{d(fg)}{dx} \, dx - \int_a^b \frac{df}{dx} g \, dx
  \]
  \[
  \int_a^b fg' \, dx = fg \bigg|_a^b - \int_a^b f'g \, dx
  \]

- Extend to vector calculus: scalar \( f \), vector \( \vec{A} \) functions

  Product rule:
  \[
  \nabla \cdot (f \vec{A}) = f (\nabla \cdot \vec{A}) + \vec{A} \cdot \nabla f
  \]

  Volume integration:
  \[
  \int_V f (\nabla \cdot \vec{A}) \, dv = \oint_S (f \vec{A}) \cdot \vec{ds} \quad \text{[Divg. thm]}
  \]
  Rearranging:
  \[
  \int_V f (\nabla \cdot \vec{A}) \, dv = \oint_S (f \vec{A}) \cdot \vec{ds} - \int_V \vec{A} \cdot \nabla f \, dv
  \]
### Miscellaneous: Some vector calculus identities

- \( \nabla \times \nabla f = 0 \) for *any* scalar function \( f \)

- \( \nabla \cdot (\nabla \times \vec{A}) = 0 \) for *any* vector field \( \vec{A} \)

- \( \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \)

- Vector field is specified up to a constant: if curl \( (\nabla \times \vec{A}) \) and divergence \( \nabla \cdot \vec{A} \) are specified.
Miscellaneous: Getting the normal to a curve

A function: \( y = f(x) \)

Vector along the tangent at some point: \( \vec{v} = (1, \frac{df}{dx}) \)

Thus \( \hat{n} \) is along \( \nabla g \). Useful for boundary conditions in Electromagnetics.
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Reference: Chapter 1 of David Griffiths: Introduction to Electrodynamics, 4rth Ed., Pearson