**EE5120 Linear Algebra: Tutorial 6, July-Dec 2018, Dr. Uday Khankhoje, EE IIT Madras**
Covers 5.3, 5.5,5.6 of GS

1. Theory of Eigenvalues and eigenvectors can be used to solve differential equations of multiple variables of the form: $\frac{d\mathbf{u}}{dt} = P\mathbf{u}$ for $\mathbf{u}(t)$, given its initial value $\mathbf{u}(0)$. If $\mathbf{u}(t) = ce^{\lambda t}\mathbf{x}$, then $\frac{d\mathbf{u}}{dt} = c\lambda e^{\lambda t}\mathbf{x}$. We can see that $\mathbf{x}$ is the eigenvector with eigenvalue $\lambda$ for the matrix $P$. Also, $\mathbf{u}(0) = c\mathbf{x} \Rightarrow \mathbf{u}(t) = ce^{\lambda t}\mathbf{x}$
   So, Any given arbitrary initial value can be expanded in terms of the eigenvectors of $P$ matrix, i.e, $\mathbf{u}(0) = \sum_{i=1}^{n} c_i\mathbf{v}_i$. Then, $\mathbf{u}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i}\mathbf{v}_i$.
   Solve the below differential equation for $\mathbf{u}(t)$
   $$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}\mathbf{u}, \quad with \quad \mathbf{u}(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$
   Is the output bounded? If not, for what values of $\mathbf{u}(0)$ will $\mathbf{u}(t)$ be bounded?

   *Hint:* $\mathbf{u}(t)$ should be bounded for bounded $\mathbf{u}(0)$.

2. The matrices $A$ and $B$ are said to be similar if there exists an invertible matrix $M$ such that $A = MBM^{-1}$.
   (a) The *identity transformation* takes every vector to itself: $Tx = x$. Find the corresponding matrix, if both the input and output bases are $\mathbf{v_1} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\mathbf{v_2} = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$. How is this matrix related to identity matrix? Are they similar?
   (b) If the transformation $T$ is reflection across the 45 degree line in the plane, find its matrix with respect to the standard basis $\mathbf{e_1} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $\mathbf{e_2} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Find the corresponding matrix when both the input and output bases are $\mathbf{v_1}$ and $\mathbf{v_2}$ as mentioned in (a). Show that these two matrices are similar by finding the matrix $M$. Give a geometrical interpretation of $M$.

3. Suppose $A$ is a $3 \times 3$ symmetric matrix with eigenvalues 0,1,2.
   (a) What properties can be guaranteed for the corresponding unit eigenvectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$?
   (b) In terms of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ describe the nullspace, left nullspace, rowspace, and columnspace of $A$.
   (c) Find a vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{v} + \mathbf{w}$. Is $\mathbf{x}$ unique?
   (d) Under what conditions on $\mathbf{b}$ does $A\mathbf{x} = \mathbf{b}$ have a solution?
   (e) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are the columns of $S$, what are $S^{-1}$ and $S^{-1}AS$?

4. Let $A$ be an $n \times n$ complex matrix. Assume $A = [\mathbf{a}_1\ \mathbf{a}_2\ ...\ \mathbf{a}_n]$, where $\mathbf{a}_i$ refers to the $i^{th}$ column of matrix $A$. Define a parameter $\mu_A$, for matrix $A$, as,
   $$\mu_A = \max_{k \neq l} \frac{|\mathbf{a}_k^H \mathbf{a}_l|}{||\mathbf{a}_k||_2 ||\mathbf{a}_l||_2}.$$
   In the literature of compressive sensing, $\mu_A$ is called the mutual coherence parameter of matrix $A$. Recall $||\mathbf{a}_i||_2 = \sqrt{\mathbf{a}_i^H \mathbf{a}_i}$, where $\mathbf{a}_i^H$ denotes hermitian (i.e., complex conjugate transpose) of $\mathbf{a}_i$, and define $B = A^H A$.
   (a) Show that $0 \leq \mu_A \leq 1$.
   (b) Denote $[B]_{i,j}$ as the $(i,j)^{th}$ entry in matrix $B$. Prove that for every $\lambda$ being an eigenvalue of $B$, there exists at least one row, say some $m^{th}$ row, of $B$ such that,
   $$\left|\lambda - [B]_{m,m}\right| \leq \sum_{p=1, p \neq m}^{n} \left|[B]_{m,p}\right|.$$
   Refer to the technique used to solve Q9) in tutorial 5 and follow a similar procedure here too. Also, the above result is independent of the information assumed that $B = A^H A$. It is true for any complex square matrix.

(c) Suppose $\mathbf{x}$ is some arbitrary $n$-length non-zero complex column vector such that $||\mathbf{x}||^2 = 1$, then prove that,

$$\lambda_{\min} \leq \mathbf{x}^H B \mathbf{x} \leq \lambda_{\max},$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of the matrix $B$. From this part onwards, the information that $B = A^H A$ is needed.

(d) Suppose all the columns of $A$ have unit norm, then prove that $\lambda_{\min}$ and $\lambda_{\max}$, as defined in (c), are bounded as,

$$\lambda_{\max} \leq 1 + (n-1)\mu_A, \text{ and}$$
$$\lambda_{\min} \geq 1 - (n-1)\mu_A.$$

Make use of the result derived in (b) to obtain the above equations.

(e) Finally, deduce that for a general vector $\mathbf{x} \neq \mathbf{0}$ and matrix $A$, the generalized result combining (c) and (d) will look like,

$$a_{\min} - a_{\max}(n-1)\mu_A \leq \frac{\mathbf{x}^H B \mathbf{x}}{||\mathbf{x}||^2} \leq a_{\max}\Big(1 + (n-1)\mu_A\Big),$$

where $a_{\min} = \min_{1 \leq k \leq n} ||\mathbf{a}_k||^2$ and $a_{\max} = \max_{1 \leq k \leq n} ||\mathbf{a}_k||^2$.

5. Define matrix $D$ as, $D = [A_1\ A_2\ ...\ A_K\ B_1\ B_2\ ...\ B_M]$, where all the sub-matrices $A_i$'s and $B_j$'s are of size $m \times n$ with $m > n$ and has unit norm columns. Let $\mu_D$ denote the mutual coherence of matrix $D$ (refer to Q 4. for definition of mutual coherence). Suppose,

$$\mathbf{y} = \sum_{k=1}^{K} A_k \mathbf{x}_k,$$

where $\mathbf{x}_k$'s are $n \times 1$ vectors such that $||\mathbf{x}_1||^2 = ||\mathbf{x}_2||^2 = ... = ||\mathbf{x}_K||^2$.

(a) Show that,

$$||A_k^H \mathbf{y}||_2 \geq \Big[1 - (Kn-1)\mu_D\Big] ||\mathbf{x}_k||_2.$$

First, try to lower bound the LHS term as $||P\mathbf{x}_k||_2 - a$, where $P$ is an hermitian matrix and $a$ is an appropriate scalar. For this, you might have to use the following fact: For any two column vectors $\mathbf{u}, \mathbf{v}$, the statement $||\mathbf{u} + \mathbf{v}||_2 \geq ||\mathbf{u}||_2 - ||\mathbf{v}||_2$ holds true. After which, incorporate the results stated as questions in Q 4 to arrive at the desired inequality equation.

(b) Now, prove that the term $||B_m^H \mathbf{y}||_2$ can be upper bounded as,

$$||B_m^H \mathbf{y}||_2 \leq Kn\mu_D ||\mathbf{x}_k||_2,$$

for some $k = 1, ..., K$.

(c) As a last step, prove that $\max_{1 \leq l \leq K} ||A_l^H \mathbf{y}||_2 > \max_{1 \leq m \leq M} ||B_m^H \mathbf{y}||_2$ can hold true if,

$$K < \frac{1}{2n}\Big(1 + \frac{1}{\mu_D}\Big).$$

The above is an important result obtained in *Greedy Algorithm* theory. The scenario has been simplified in this question to make the entire derivation straight-forward. However, the final result provides a sufficient condition under which a particular greedy algorithm will be able to solve a special type of compressive sensing problem.

6. Consider two adjoining cells separated by a permeable membrane, and suppose that a fluid flows from the first cell to the second one at a rate (in milliliters per minute) that is numerically equal to three times the volume (in milliliters) of the fluid in the first cell. It then flows out of the second cell at a rate (in milliliters per minute) that is numerically equal to twice the volume in the second cell. Let $x_1(t)$ and $x_2(t)$ denote the volumes of the fluid in the first and second cells at time $t$, respectively. Assume that, initially, the first cell has 40 milliliters of fluid, while the second one has 5 milliliters of fluid. Find the volume of fluid in each cell at time t.

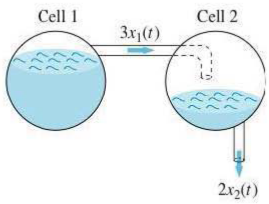


Figure 1: Figure for Q.6

7. (a) Prove that $C(A), N(A), C(A^H)$ and $N(A^H)$ are the fundamental spaces in complex case, i.e, give their properties and derive them. Verify for the matrix

$$A = \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix}$$

   (b) Prove that determinant of a Hermitian matrix is real.

8. The Harmonic Oscillator is shown in figure 2, where a particle is constrained by some kind of forces like in the case of atoms in solids. Find the energy values that can be taken by

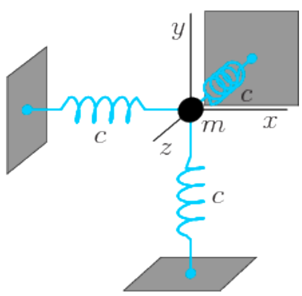


Figure 2: Classical picture of an harmonic oscillator

particle by solving the following eigen equation:

$$\mathbf{H}\psi = E\psi$$

where $\psi$ is the eigen function which gives the probability of finding the particle at (x,y,z), $E$ is the corresponding eigen energy value and $\mathbf{H}$ is the Hamiltonian given as below,

$$\mathbf{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{c}{2}(x^2 + y^2 + z^2)$$

*Hint:* Take $\psi(x,y,z) = \psi_x(x)\psi_y(y)\psi_z(z)$. Eigen functions are continuous equivalent of eigenvectors.