EE5120 Linear Algebra: Tutorial Test 6, 05.11.18A

Give your answers in the space provided. No calculators or smartphones allowed. Please take a few minutes to read the questions carefully and answer (briefly) only what is asked.

Roll: No:________________ NAME:________________________ Time: 20 mins

Useful fact from tutorial: For any matrix A, for every eigenvalue of A there exists a row (call it the mth row) such that $|\lambda - A_{mm}| \leq \sum_{i=1,i\neq m}^{n} |A_{mi}|$.

$\boxed{3}$ 1. Consider a matrix A of the form $A = B^H B$ and an arbitrary vector u. Express $A^n u$ purely in terms of eigenvalues/vectors of A. Justify your steps.

> **Solution:** Spectral theorem holds for A since it is a Hermitian matrix, thus $A = U\Lambda U^H = \sum_i \lambda_i u_i u_i^H$ [1].
> This leads to $A^n = U\Lambda^n U^H = \sum_i \lambda_i^n u_i u_i^H$ [1].
> Finally $A^n u = \sum_i \lambda_i^n u_i (u_i^H u)$ [1].

$\boxed{3}$ 2. Consider the usual 3 dimensional space characterized by cartesian coordinates, C. Now, consider another coordinate system D derived from C by rotation about the z axis by 30° (place your right thumb along the z-axis and rotate counter-clockwise). Find the 3×3 matrix that will convert the coordinates of a vector expressed in C's frame to the coordinates in D's frame. Draw diagrams where necessary.

> **Solution:** A vector in C frame has basis vectors $[c_1, c_2, c_3]$ and coordinates $\alpha = [\alpha_1, \alpha_2, \alpha_3]$, and in D has basis vectors $[d_1, d_2, d_3]$ and coordinates $\beta = [\beta_1, \beta_2, \beta_3]$. Since the point remains the same, but only the representation changes, (i.e. identity transformation T) we can say that : $T([c_1\, c_2\, c_3]) = [d_1\, d_2\, d_3]M$ [1]. We take the c_i's to be the canonical basis and going one by one we see that $M = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ [1]. To get the final answer, i.e. the β's given the α's we only need to do $\beta = M\alpha$. [1]
> As an example, take the point in C frame as $\alpha = (\cos\theta, \sin\theta, 0)$. We know that this lies on the new x axis after rotation, i.e. in D frame it should be $\beta = (1, 0, 0)$. You can verify that $\beta = M\alpha$.

$\boxed{4}$ 3. Denote the rows of a square matrix A by A_i. This matrix has a special property that the 1-norm of each row is given as $\|A_i\|_1 = i$, for $i = 1 \ldots n$. Can you estimate a bound on the maximum and minimum possible eigenvalues of A? Recal the defn $\|x\|_1 = \sum_i |x_i|$

> **Solution:** We can interpret the useful fact geometrically as follows: Let A_{mm} be the centre of a circle along the x-axis, whose radius is $\sum_{i=1,i\neq m}^{n} |A_{mi}|$, thus λ lies within this circle as per the given relation [2]. Thus the maximum possible eigenvalue can happen when λ is at the extreme right point on the biggest possible circle, i.e. $\lambda = A_{mm} + \sum_{i=1,i\neq m}^{n} |A_{mi}| = \|A_m\|_1$. Thus $\lambda_{\max} = \max\|A_i\|_1 = n$ [1]. Similarly, the minimum can happen when it is at the extreme left end of the biggest circle, $\lambda_{\min} = -\max\|A_i\|_1 = -n$ [1]. Draw and see this to get clarity.