EE5120 Linear Algebra: Tutorial 5, July-Dec 2018, Dr. Uday Khankhoje, EE IIT Madras

Covers Ch 4.1,4.2,5.1,5.2 of GS

1. Find the value of k in each of the following cases so that it satisfies the corresponding equation.

   (a)
$$\det\begin{bmatrix} 3a & 3b & 3c \\ 3p & 3q & 3r \\ 3x & 3y & 3z \end{bmatrix} = k \det\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}.$$

   (b)
$$\det\begin{bmatrix} 2a & 2b & 2c \\ 3p+5x & 3q+5y & 3r+5z \\ 7x & 7y & 7z \end{bmatrix} = k \det\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}.$$

   (c)
$$\det\begin{bmatrix} p+x & q+y & r+z \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{bmatrix} = k \det\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}.$$

   Hint: Use properties of determinants.

2. Prove each of the following statements.

   (a) Two square matrices A and B, of same size, are said to be similar, if there exists an invertible matrix P of same size as that of A (or B) such that $A = PBP^{-1}$. Prove that determinant of A and B are same.

   (b) Let M be an $n \times n$ matrix with complex-valued entries in it. Matrix M^* refers to the complex conjugate of matrix M, i.e., if $[M]_{i,j}$ is the $(i,j)^{\text{th}}$ of matrix M, then $(i,j)^{\text{th}}$ element of the matrix M^* equals $[M]^*_{i,j}$. Show that det(M^*) = $\left(\det(M)\right)^*$.

   (c) Determinant of the matrix $A + tI$, where $t \neq 0$, I is an $n \times n$ identity matrix and,
$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & a_0 \\ -1 & 0 & 0 & \dots & a_1 \\ 0 & -1 & 0 & \dots & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-1} \end{bmatrix},$$
   is equal to $t^n + \sum\limits_{i=0}^{n-1} a_i t^i$.

   Hint: Use the properties of determinants for (a), and the definition of determinants for (b) and (c).

3. (a) Let $\mathsf{L} : \mathbb{M}_{n\times n} \to \mathbb{R}^{1\times n}$ be a linear transformation, with $\mathbb{M}_{n\times n}$ being the set of all $n \times n$ matrices, defined as $\mathsf{L}(P) = \mathbf{x}^T AP - \mathbf{x}^T P$, for any $P \in \mathbb{M}_{n\times n}$. Here, A is some fixed $n \times n$ symmetric matrix and $\mathbf{x}$ is some fixed $n \times 1$ column vector. If it is given that all invertible $n \times n$ matrices from $\mathbb{M}_{n\times n}$ map to $\mathbf{0}^T \in \mathbb{R}^{1\times n}$ under the transformation L, can you comment about at least one eigenvalue and one eigenvector of matrix A?

   (b) Let $H = I - 2\mathbf{u}\mathbf{u}^T$, where I is $n \times n$ identity matrix and $\mathbf{u}$ is an $n \times 1$ column vector such that $\mathbf{u}^T\mathbf{u} = 1$. Can you comment on at least two eigenvalues and corresponding eigenvectors of H?

(c) Let $A = \begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix}$, where $b \neq 0, c \neq 1$ and b, c are real numbers. Compute eigenvalues and eigenvectors of the matrices A and $B = \begin{bmatrix} A & O \\ O & A \end{bmatrix}$, where O is a 2×2 all-zero matrix.

Hint: Use definition of eigenvalue/eigenvector of a square matrix.

4. Show that the sum of eigenvalues of a matrix is given by its trace, and that the product of eigenvalues is given by its determinant.

5. (i) Given that $A\mathbf{x} = \lambda\mathbf{x}$, prove the following:
   (a) λ^2 is an eigenvalue of A^2,
   (b) λ^{-1} is an eigenvalue of A^{-1},
   (c) $\lambda + 1$ is an eigenvalue of $A + I$.

   (ii) A 3×3 matrix B is known to have eigenvalues $0, 1, 2$. This information is enough to find three of these:
   (a) the rank of B,
   (b) the determinant of $B^T B$,
   (c) the eigenvalues of $B^T B$, and
   (d) the eigenvalues of $(B + I)^{-1}$.

6. Prove that two $n \times n$ matrices are equal if all their eigenvalues and their corresponding eigenvectors are equal, and the matrices have n linearly independent eigenvectors.

7. The powers A^k of this matrix A approaches a limit as $k \to \infty$:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}, \quad A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix}, \quad \textit{and} \quad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$$

The matrix A^2 is halfway between A and A^∞. Explain why $A^2 = \frac{1}{2}(A + A^\infty)$ from the eigenvalues and eigenvectors of these three matrices. *Hint:* Use the fact from Q6.

8. Consider a matrix A of size $n \times n$. If A has $(n_1 + 1)$ distinct eigen values and one of them is repeated n_2 number of times, satisfying $n_1 + n_2 = n$, then derive a condition that can ensure the diagonalizability of A.

Hint: Find the condition in terms of rank of some matrix.

9. An $n \times n$ matrix M is said to be 'Markov matrix' if all it's entries are non-negative and the sum of the entries of each column is 1. If $\{\lambda_i\}$ are the eigen values of M and M is a real matrix, prove the followings
   (a) $\lambda_1 = 1$ is always an eigen value of M.
   (b) $|\lambda_i| \leq 1 \;\; \forall i \in \{1, \dots n\}$.

Hint: (b) Use the fact that the matrices A and A^T have same eigen values.

10. If $p(\lambda) = \prod_{i=1}^{m} (\lambda - \lambda_i)$ is the characteristic polynomial of a matrix A with distinct eigen values, then find the characteristic polynomial of the matrix $A^n - kI$, where I is the identity matrix of appropriate dimension and $k, n \in \mathbb{R}$.

11. Describe how you might try to build a solution of a difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$, ($k = 0, 1, 2, ...$), if you were given the initial $\mathbf{x}_0$ and this vector did not happen to be an eigenvector of A. Assume A is an $p \times p$ matrix with all its p eigenvectors being linearly independent.

12. Consider a linear operator T, $\mathsf{T} : \mathcal{V} \longrightarrow \mathcal{V}$, where $\mathcal{V}$ is a vector space and let $\mathcal{E}_\lambda = \{\mathbf{x} \big| T(\mathbf{x}) = \lambda\mathbf{x}\}$ (called the eigen space of λ). Prove that $\mathcal{E}_\lambda$ is a subspace of $\mathcal{V}$.

13. An elastic object in the xy plane with a circular boundary $x^2 + y^2 = 1$ is stretched so that a point P(x_1, y_1) goes over into the point Q(x_2, y_2) given by

$$b = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix}$$

Find the principal directions, that is the directions of the position vector d_1 of P for which the direction of the position vector d_2 of Q is the same or exactly opposite. What shape does the boundary circle take under the deformation?

14. Let

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

(a) Find all eigenvalues and corresponding eigenvectors.

(b) Find a nonsingular matrix P such that $D = P^{-1}AP$ is diagonal, and P^{-1}.

(c) Find A^6 and $f(A)$, where $f(t) = t^4 - 3t^3 - 6t^2 + 7t + 3$.

(d) Find a real cube root of B, that is, a matrix B such that $B^3 = A$ and B has real eigenvalues. Assume B is diagonalizable.