EE5120 Linear Algebra: Tutorial 4, July-Dec 2018, Dr. Uday Khankhoje, EE IIT Madras
Covers Ch 3.1,3.2,3.3,3.4 of GS

1. Given a matrix P that satisfies $P^2 = P$ and $P^T = P$. Using these facts, prove that Pb is the projection of b onto the column space of P.

   Hint: Use appropriate orthogonality relations of subspaces

2. (a) Consider the system of linear equations $A\mathbf{x} = \mathbf{b}$, where A is a full column rank matrix, $\mathbf{x}$ is an n-length vector and $\mathbf{b}$ is an m-length vector. What is the least squares (LS) solution to the above system? Prove that the error in the estimate is in the left null-space of A.

   (b) Once again consider $A\mathbf{x} = \mathbf{b}$, with $A = \begin{bmatrix} 1 & 3 & -1 & 3 & 2 \\ 2 & -1 & 0 & 1 & 0 \end{bmatrix}^T$ and $\mathbf{b} = \begin{bmatrix} -1 & 4 & 1 & 2 & 1 \end{bmatrix}^T$. Can you find a solution to the given system of linear equations using Gauss elimination? Can you find a LS solution? Verify (a) for this example. Further, determine the left null space of A and verify actually whether the LS error lies in the left null space of A or not, by expressing the error in terms of the basis of left null space of A.

3. Consider the QR decomposition of a matrix A as shown below. Matrix on the LHS is A. On the RHS, first matrix is Q and is multiplied with R. Fill in the blanks.

$$\begin{bmatrix} 1 & \text{----} & -1 & \text{----} \\ 1 & \text{----} & 2 & \text{----} \\ -1 & \text{----} & 3 & \text{----} \\ 1 & \text{----} & 1 & \text{----} \end{bmatrix} = \begin{bmatrix} \text{----} & 0 & \text{----} & \frac{6}{\sqrt{72}} \\ \text{----} & \frac{1}{\sqrt{2}} & \text{----} & -\frac{4}{\sqrt{72}} \\ \text{----} & \frac{1}{\sqrt{2}} & \text{----} & \frac{4}{\sqrt{72}} \\ \text{----} & 0 & \text{----} & \frac{2}{\sqrt{72}} \end{bmatrix} \begin{bmatrix} \text{----} & 0 & \text{----} & 0.5 \\ \text{----} & \sqrt{2} & \text{----} & 2.1213 \\ \text{----} & 0 & \text{----} & 0.5 \\ \text{----} & 0 & \text{----} & 1.4815 \end{bmatrix}.$$

4. (a) Prove that the trace of $P = \mathbf{a}\mathbf{a}^T/\mathbf{a}^T\mathbf{a}$—which is the sum of its diagonal entries—always equal 1.

   (b) Is the projection matrix P invertible? Why or why not?

5. (a) Show that length of $A\mathbf{x}$ equals the length of $A^T\mathbf{x}$ iff $AA^T = A^TA$.

   (b) If Q_1 and Q_2 are orthogonal matrices in $\mathbb{R}^{2\times 2}$, so that $Q^TQ = I$, show that Q_1Q_2 is also orthogonal. If Q_1 is rotation through θ, and Q_2 is rotation through ϕ, what is Q_1Q_2? Can you find the trigonometric identities for $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$ in the matrix multiplication Q_1Q_2?

6. (a) Teacher taught kinematics in the class that, $s = ut + \frac{1}{2}at^2$. She asked all the students to find out what is the earth's acceleration due to gravity by doing a experiment at home. What would you do to get an accurate value.

   (b) Given a system with input and output relation: $y = ax + bx^2$. To find the coefficients, an experiment is done and data is shown below. Find the coefficients?

x	20	21	22	23	24	25	26
y	2396	2562	2727	2904	3070	3254	3427

   Hint: Use Least square error curve fit.

7. Let $\mathbb{P}_3(t)$ be the vector space of polynomials at degree at most equal to 3, defined with inner product $\big(f(t), g(t)\big) = \int_{-1}^{1} f(t)g(t)dt$, for any two functions $f(t), g(t) \in \mathbb{P}_3$. Apply the Gram-Schmidt procedure to $\{1, t, t^2, t^3\}$ to find an orthonormal basis $\{f_0, f_1, f_2, f_3\}$.

8. Let $\bar{x} = \frac{1}{n}(x_1 + ... + x_n)$ and $\bar{y} = \frac{1}{n}(y_1 + ... + y_n)$. Show that the least-squares line for the data $(x_1, y_1), ..., (x_n, y_n)$ must pass through $(\bar{x}, \bar{y})$. That is, show that $\bar{x}$ and $\bar{y}$ satisfy the linear equation $\bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}$.

9. Suppose $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ is an orthonormal basis of $\mathcal{V}$. Prove
(a) For any $\mathbf{u} \in \mathcal{V}$, we have $\mathbf{u} =< \mathbf{u}, \mathbf{e}_1 > \mathbf{e}_1 + < \mathbf{u}, \mathbf{e}_2 > \mathbf{e}_2 + ... + < \mathbf{u}, \mathbf{e}_n > \mathbf{e}_n$.
(b) $< a_1\mathbf{e}_1 + ... + a_n\mathbf{e}_n, b_1\mathbf{e}_1 + ... + b_n\mathbf{e}_n >= a_1b_1 + a_2b_2 + ...a_nb_n$.
(c) For any $\mathbf{u}, \mathbf{v} \in \mathcal{V}$, we have $< \mathbf{u}, \mathbf{v} >=< \mathbf{u}, \mathbf{e}_1 >< \mathbf{v}, \mathbf{e}_1 > + ... + < \mathbf{u}, \mathbf{e}_n >< \mathbf{v}, \mathbf{e}_n >$

10. Consider the function $f : \mathbb{R} \to \mathbb{R}$. Sample input,output pairs of the function are given as $(1, 1), (2, 3), (3, 1)$
   (a) Find the straight line $y = mx$, which has least mean squared error at the given sample points, using calculus.
   (b) Find the same line (find m), by writing a system of linear equations then projecting a vector onto the columnspace of system matrix. Find the vector that is being projected and find the vectorspace onto which that vector is projected.
   (c) Compare the data given in (a) and vector\vectorspace used for projection in (b). Convince yourself that the picture used in (a) and that in (b) are not same.