EE5120 Linear Algebra: Tutorial Test 4, 04.10.18A

Give your answers in the space provided. Please take a few minutes to read the questions carefully and answer (briefly) only what is asked.

Roll: No:________________ NAME:________________________ Time: 15 mins

$\boxed{10}$ 1. You are given a set of polynomials $\{1, t, t^2\}$ that are defined over the interval $(0, \infty)$, and the definition of inner product is modified as: $(f, g) = \int_0^\infty f(t)g(t)w(t)\,dt$, where $w(t) = \exp(-t)$ (Notice the change, earlier $w(t) = 1$). As always, squared length of a function is $\|f\|^2 = (f, f)$.
(a) What is the length of each of the functions in the given set?
(b) Convert the given set of polynomial functions to an **orthogonal** set using Gram Schmidt (i.e. it is sufficient that the set you produce is orthogonal – need not be orthonormal). Additional information given is that $\int_0^\infty t^n e^{-t} dt = (n)(n-1)(n-2)\cdots 1 = n!$.

If you can solve this problem you have just learnt how to produce Laguerre polynomials from standard polynomials. By changing the form of $w(t)$, many different kinds of orthogonal polynomials can be generated, e.g. Legendre, Hermite, Laguerre, etc.

Solution:

1. Squared lengths of given set: $\|1\|^2 = 1$, $\|t\|^2 = \int_0^\infty t^2 e^{-t} dt = 2$, and $\|t^2\|^2 = \int_0^\infty t^4 e^{-t} dt = 24$.
2. First polynomial will be $f_1(t) = 1/\|1\|$ and since $\|1\| = \int_0^\infty e^{-t} dt = 1$, $f_1(t) = 1$.
3. Second polynomial is $f_2(t) = t - \frac{(t, f_1(t))}{(f_1(t), f_1(t))} f_1(t) = t - (\int_0^\infty te^{-}t\,dt)/1 \cdot 1 = t - 1$
4. Third polynomial is $f_3(t) = t^2 - \frac{(t^2, f_2(t))}{(f_2(t), f_2(t))} f_2(t) - \frac{(t^2, f_1(t))}{(f_1(t), f_1(t))} f_1(t)$

   Note that $(f_2(t), f_2(t)) = \int (t^2 - 2t + 1)e^{-t} dt = 2 - 2 + 1 = 1$.
   This gives: $f_3(t) = t^2 - (\int (t^3 - t^2)e^{-t} dt)(t-1) - (\int (t^2)e^{-t} dt)1 = t^2 - 4(t-1) - 2 = t^2 - 4t + 2$

These f_i are our orthogonal polynomials.