EE5120 Linear Algebra: Tutorial 3, July-Dec 2018, Dr. Uday Khankhoje, EE IIT Madras

Covers Ch 2.4, 2.6 of GS

1. (a) Find the dimension and a basis for the four fundamental subspaces for

$$(i)\, A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \qquad (ii)\, U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Without computing A, find bases for the four fundamental subspaces:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(c) Without multiplying matrices, find bases for the row and column spaces of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

How do you know from these shapes that A is not invertible?

Hint:
(a) Null spaces for A^T is left null space of A.
(b) A is of the form LU and pivot elements are non-zero. So, Get $C(A)$, Left null spaces from L and $R(A)$, Null space from U.
(c) See the number of pivot elements in both matrices, as calculate same as (b).

Solution:

(a)

$$(i) \quad C(A) : r = 2, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; N(A) : n - r = 2, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$R(A) : r = 2, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \; Left\,Null(A) : m - r = 1, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$(ii) \quad C(U) : r = 2, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; N(U) : n - r = 2, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$R(U) : r = 2, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \; Left\,Null(U) : m - r = 1, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(b)

$$R(A): \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\0\\1\\2\end{bmatrix}, N(A): \begin{bmatrix}0\\-1\\-2\\1\end{bmatrix}$$

$$C(A): \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, Left\,Null(U): 0$$

(c) If $A_{m\times p} = B_{m\times n}C_{n\times p}$, and $Rank(B) = Rank(C) = n$. Then, $C(A) = C(B)$, $N(A^T) = N(B^T)$, $R(A) = R(C)$ and $N(A) = N(C)$.
Rank of both matrices are 2. So,

$$R(A): \begin{bmatrix}3\\0\\3\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}, C(A): \begin{bmatrix}1\\4\\2\end{bmatrix}, \begin{bmatrix}2\\5\\7\end{bmatrix}$$

$Rank(BC) \le min(Rank(B), Rank(C)) \Rightarrow Rank(A) \le 2$. So, Not invertible.

2. Given a rectangular matrix, prove that (a) if it has full row rank, then (AA^T) is invertible, (b) if it has full column rank, then (A^TA) is invertible. Finally, (c) prove that the left and right inverse of a square invertible matrix are identical.

Hint: Think of the sizes of the null spaces of (A^TA) and (AA^T).

Solution: Recall these basic facts: (i) if A has full row rank, then $m = r$ and the left null space which is of dimension $m - r$, is empty, i.e. $N(A^T) = \{\mathbf{0}\}$, similarly, (ii) when A has full column rank, then $n = r$ and the null space is empty, i.e. $N(A) = \{\mathbf{0}\}$.
Also note that both (A^TA) and (AA^T) are square and proving their invertibility amounts to proving that their null spaces are empty.

(a)
- Let us assume that there is a non-zero vector $\mathbf{y}$ such that $AA^T\mathbf{y} = \mathbf{0}$.
- Left multiply both sides by $\mathbf{y}^T$, giving $\mathbf{y}^TAA^T\mathbf{y} = \|A^T\mathbf{y}\|^2 = 0$, leading to $A^T\mathbf{y} = 0$.
- $\implies \mathbf{y}$ is in $N(A^T)$. But, as noted earlier for a full row rank matrix, $N(A^T) = \{\mathbf{0}\}$.
- Thus we have a contradiction and $\mathbf{y}$ must be $\mathbf{0}$ and AA^T is invertible.

(b) Mimic the above proof starting by assuming a non-zero vector x such that $A^TA\mathbf{x} = \mathbf{0}$, and use the fact that $N(A) = \{\mathbf{0}\}$ to arrive at a contradiction.

(c)
- Since the matrix, A is invertible, so is A^T. A has independent rows and independent columns. Hence both $(AA^T)^{-1}$ and $(A^TA)^{-1}$ are invertible.
- Left inverse of A is $(A^TA)^{-1}A^T = A^{-1}(A^T)^{-1}A^T = A^{-1}I = A^{-1}$.
- Right inverse of A is $A^T(AA^T)^{-1} = A^T(A^T)^{-1}A^{-1} = IA^{-1} = A^{-1}$.

3. Given that a set of k vectors, $S_k = \{v_1, v_r, \ldots, v_k\}$ is linearly independent. Let us expand

this set by adding a non-zero vector v_{k+1} that is orthogonal to all elements of S_k. Prove that the resulting set is also linearly independent.

Hint: Use the definition of linear independence in matrix language.

Solution: A is the matrix formed with v_is, $i \in [1, k]$ as the columns. Since S_k is linearly independent, $Ax = 0$ is possible only for $x = 0$ (by defn). Now consider the matrix B formed by appending v_{k+1} as a new column to A, i.e. $B = [A\, v_{k+1}]$. To prove linear independence of S_{k+1}, we must show that $Bw = 0$ only if $w = 0$. Rewrite as follows, with $w = [x\, y]^T$:
$Bw = [A\, v_{k+1}][x\, y]^T = Ax + v_{k+1}y = 0$, where $x \in \mathbf{R}^k$, $y \in \mathbf{R}$. Now take an inner product with v_{k+1} by left multiplying by v_{k+1}^T to get $0 + \|v_{k+1}\|^2 y = 0$, i.e. $y = 0$ regardless of the value of x. In other words, $Bw = 0$ collapses to $Ax = 0$, which we know is possible only if $x = 0$. Putting these two facts together we get that $bw = 0$ happens only when $w = 0$, and thus we have proved the linear independence of the new set of vectors.

4. (a) If A is square, (i) show that the nullspace of A^2 contains the nullspace of A. (ii) Show also that the column space of A^2 is contained in the column space of A.
   (b) If $AB = 0$, prove that $rank(A) + rank(B) \leq n$, where A is a $m \times n$ matrix.

Solution:

(a) (i) Let $\mathbf{x} \in N(A)$, $\Rightarrow A\mathbf{x} = \mathbf{0}$. Multiplying A on both sides, $A^2\mathbf{x} = A\mathbf{0} = \mathbf{0}$. So, $\mathbf{x} \in N(A) \Rightarrow \mathbf{x} \in N(A^2)$. Hence, $N(A) \subseteq N(A^2)$
(ii) It can be seen that columns of A^2 are linear combinations of columns of A.
So, $C(A^2) \subseteq C(A)$.

(b) Given $AB = 0 \Rightarrow R(A)$ are orthogonal to $C(B)$. Rows of A and columns of B are of length n. So, we can have maximum of $'n'$ orthogonal vectors. For $R(A)$ to be orthogonal to $C(B)$, select $'p'$ vectors from $'n'$ orthogonal vectors for $R(A)$ and selecting $'q'$ vectors from the remaining vectors $'n - p'$, such that $p + q \leq n$. Also, Rank of a matrix = dimension of row space of matrix = dimension of column space of matrix.
So, $rank(A) + rank(B) = p + q \leq n$.

5. Find the matrix representation for each of the following linear transformations. Also, say if the transformation is invertible or not just by looking at the matrix representation. Justify your answer.
   (a) Let $\mathbb{M}_2$ be the vector space of 2×2 real finite valued matrices, having an ordered basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. The linear transformation given is $\mathsf{L} : \mathbb{M}_2 \to \mathbb{R}$, and is defined as, $\mathsf{L}(A) = \text{trace}(A)$, where $A \in \mathbb{M}_2$.
   (b) Consider the linear transformation F on $\mathbb{R}^2$ defined by $\mathsf{F}(x, y) = (5x - y, 2x + y)$ and the following bases of $\mathbb{R}^2$: $\mathcal{E} = (e_1, e_2) = ((1, 0), (0, 1))$ and $\mathcal{S} = (u_1, u_2) = ((1, 4), (2, 7))$. Find the matrix A that represents F in the basis $\mathcal{E}$. Also, find the matrix B that represents F in the basis $\mathcal{S}$.

Solution:

(a) We have the following:

$$\begin{aligned}
\mathsf{L}\Big(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\Big) &= \text{trace}\Big(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\Big) = 1\\
\mathsf{L}\Big(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\Big) &= \text{trace}\Big(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\Big) = 0\\
\mathsf{L}\Big(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\Big) &= \text{trace}\Big(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\Big) = 0\\
\mathsf{L}\Big(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\Big) &= \text{trace}\Big(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\Big) = 1
\end{aligned}$$

Thus, the matrix representation is $L = [1\,0\,0\,1]$. It is not a square matrix and hence not invertible.

(b)

$$A = \begin{bmatrix}5 & -1\\2 & 1\end{bmatrix}.$$

To find the matrix B, find the coordinates of $\mathsf{F}(\mathbf{u}_1) = \mathsf{F}\Big((1,4)\Big)$ and $\mathsf{F}(\mathbf{u}_2) = \mathsf{F}\Big((2,7)\Big)$ relative to the basis $\mathcal{S}$. This may be done by first finding the coordinates of an arbitrary vector (a,b) in $\mathbb{R}^2$ relative to the basis $\mathcal{S}$. We have,

$$\begin{aligned}
&(a,b) = x(1,4) + y(2,7) = (x+2y, 4x+7y)\\
\Rightarrow &x + 2y = a \text{ and } 4x + 7y = b.
\end{aligned}$$

Solve for x and y in terms of a and b to get $x = -7a + b$ and $y = 4a - b$. Then,

$$(a,b) = (-7a+2b)\mathbf{u}_1 + (4a-b)\mathbf{u}_2.$$

Then,

$$\mathsf{F}(\mathbf{u}_1) = \mathsf{F}(1,4) = (1,6) = 5\mathbf{u}_1 - 2\mathbf{u}_2,$$

and

$$\mathsf{F}(\mathbf{u}_2) = \mathsf{F}(2,7) = (3,11) = \mathbf{u}_1 + \mathbf{u}_2.$$

Thus, the matrix representation is,

$$B = \begin{bmatrix}5 & 1\\-2 & 1\end{bmatrix}.$$

Further, since both matrices A and B are full rank square matrices, they are invertible.

6. Let $\mathcal{V}$ be a vector space and $\mathsf{T} : \mathcal{V} \to \mathcal{V}$ be a linear transformation. Suppose $\mathbf{x} \in \mathcal{V}$ is such that $\mathsf{T}^k(\mathbf{x}) = \mathbf{0}$, $\mathsf{T}^m(\mathbf{x}) \neq \mathbf{0}$, $\forall\ \ 1 \le m < k$ and $k > 1$, then prove that the set of vectors $\{\mathbf{x}, \mathsf{T}(\mathbf{x}), \mathsf{T}^2(\mathbf{x}), ..., \mathsf{T}^{k-1}(\mathbf{x})\}$ is linearly independent.

Solution: Given $k > 1$. Thus, $\mathsf{T}(\mathbf{x}) \neq \mathbf{0} \Rightarrow \mathbf{x} \neq \mathbf{0}$. Since $\mathsf{T}^k(\mathbf{x}) = \mathbf{0}$, for all $p \geq 1$,

$$\mathsf{T}^{k+p}(\mathbf{x}) = \mathsf{T}^p\Big(\mathsf{T}^k(\mathbf{x})\Big) = \mathsf{T}^p(\mathbf{0}) = \mathbf{0}. \tag{1}$$

Assume that $\{\mathbf{x}, \mathsf{T}(\mathbf{x}), \mathsf{T}^2(\mathbf{x}), ..., \mathsf{T}^{k-1}(\mathbf{x})\}$ is linearly dependent. Then,

$$a_1\mathbf{x} + a_2\mathsf{T}(\mathbf{x}) + ... + a_k\mathsf{T}^{k-1}(\mathbf{x}) = 0,$$

with not all a_i's being zero, i.e., some a_is are not equal to zero. Now, consider the following:

$$\begin{aligned}
&\mathsf{T}^{k-1}\Big(a_1\mathbf{x} + a_2\mathsf{T}(\mathbf{x}) + ... + a_k\mathsf{T}^{k-1}(\mathbf{x})\}\Big) = \mathsf{T}^{k-1}(\mathbf{0})\\
\Rightarrow &a_1\mathsf{T}^{k-1}(\mathbf{x}) + a_2\mathsf{T}^{k}(\mathbf{x}) + a_3\mathsf{T}^{k+1}(\mathbf{x}) + ... + a_k\mathsf{T}^{2(k-1)}(\mathbf{x}) = \mathbf{0}\\
\Rightarrow &a_1\mathsf{T}^{k-1}(\mathbf{x}) + \mathbf{0} + \mathbf{0} + ... + \mathbf{0} = \mathbf{0}.
\end{aligned}$$

The above result is a consequence of equation (1) and other given information. Since $\mathsf{T}^{k-1}(\mathbf{x}) \neq \mathbf{0}$, $a_1 = 0$. Now,

$$\begin{aligned}
&\mathsf{T}^{k-2}\Big(a_1\mathbf{x} + a_2\mathsf{T}(\mathbf{x}) + ... + a_k\mathsf{T}^{k-1}(\mathbf{x})\}\Big) = \mathsf{T}^{k-2}(\mathbf{0})\\
\Rightarrow &a_1\mathsf{T}^{k-2}(\mathbf{x}) + a_2\mathsf{T}^{k-1}(\mathbf{x}) + a_3\mathsf{T}^{k}(\mathbf{x}) + ... + a_k\mathsf{T}^{2k-3}(\mathbf{x}) = \mathbf{0}\\
\Rightarrow &\mathbf{0} + a_2\mathsf{T}^{k-1}(\mathbf{x}) + \mathbf{0} + ... + \mathbf{0} = \mathbf{0}.
\end{aligned}$$

Again, since $\mathsf{T}^{k-1}(\mathbf{x}) \neq \mathbf{0}$, we get $a_2 = 0$. On repeating this procedure, we get $a_i = 0, \forall i = 1, 2, ..., k$, which is contradicting to the initial assumption. Hence, the initial assumption of the set $\{\mathbf{x}, \mathsf{T}(\mathbf{x}), \mathsf{T}^2(\mathbf{x}), ..., \mathsf{T}^{k-1}(\mathbf{x})\}$ being linearly dependent is incorrect. Thus, the above set is linearly independent.

7. Prove that,
   (a) A linear transformation $\mathsf{L} : \mathcal{V} \longrightarrow \mathcal{W}$ is invertible if and only if the matrix representation for L is square and its null-space has only all-zero element.
   (b) L^{-1} is also a linear transformation and $(\mathsf{L}^{-1})^{-1} = \mathsf{L}$.

Solution:

(a) **part 1:** Assume that L is a square matrix and its null space has only the zero vector. *To prove:* L is invertible, i.e., L is one-to-one and onto. **one-to-one:** Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$ s.t. $\mathsf{L}(\mathbf{v}_1) = \mathsf{L}(\mathbf{v}_2)$. This implies, $\mathsf{L}(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0} \Rightarrow \mathbf{v}_1 - \mathbf{v}_2 \in$ null space of L. But null space of L has only all-zero vector. So, $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0} \Rightarrow \mathbf{v}_1 = \mathbf{v}_2$. This proves one-to-one relation. **onto:** It is required to prove that the range space of L equals the output vector space $\mathcal{W}$. Let $\mathbf{u}_1, ..., \mathbf{u}_n$ be basis of $\mathcal{V} \Rightarrow \dim(\mathcal{V}) = n$. Now, for some scalars $a_1, ..., a_n$,

$$\sum_{i=1}^{n} a_i\mathsf{L}(\mathbf{u}_i) = \mathbf{0} \Rightarrow \mathsf{L}\Big(\sum_{i=1}^{n} a_i\mathbf{u}_i\Big) = \mathbf{0} \Rightarrow \sum_{i=1}^{n} a_i\mathbf{u}_i \in \text{ null space of } \mathsf{L}.$$

This further implies, $\sum_{i=1}^{n} a_i \mathbf{u}_i = \mathbf{0} \Rightarrow a_1 = a_2 = ... = a_n = 0$ as $\mathbf{u}_i$'s are basis vectors. Hence, we now showed that the set of vectors $\mathcal{S} = \left\{ \mathsf{L}(\mathbf{u}_1), ..., \mathsf{L}(\mathbf{u}_n) \right\}$ is linearly independent in $\mathcal{W}$, infact, in the range space of L. Let $\mathbf{b} \in$ range space of L. Then, $\exists \mathbf{u} \in \mathcal{V}$ s.t. $\mathsf{L}(\mathbf{u}) = \mathbf{b}$. But $\mathbf{u}$ will have a unique basis expansion; let it be $\mathbf{u} = \sum_{i=1}^{n} b_i \mathbf{u}_i$. Then, we get,

$$\mathbf{b} = \mathsf{L}(\mathbf{u}) = \mathsf{L}\Big(\sum_{i=1}^{n} b_i \mathbf{u}_i \Big) = \sum_{i=1}^{n} b_i \mathsf{L}(\mathbf{u}_i).$$

This is true for any $\mathbf{b}$ in range space of L. Thus, the set $\mathcal{S}$ is the basis for the range space of L which implies dimension of range space of L equals n. L being a square matrix implies $\dim(\mathcal{V}) = \dim(\mathcal{W}) = n$. Since range space of L is a sub-space of $\mathcal{W}$ having same dimension, range space of L must be equal to $\mathcal{W}$. Hence, proved.

part 2: Now assume that L is an invertible transformation, i.e., it is one-to-one and onto. We know that $\mathsf{L}(\mathbf{0}) = \mathbf{0}$. Because of one-to-one nature no other element in $\mathcal{V}$ will map to $\mathbf{0}$ in $\mathcal{W}$, thus null space of L has only the all-zero vector. Further, onto implies range space of L equals $\mathcal{W} \Rightarrow \dim(\mathcal{W})$ equals that of range space of L. But by proof in part 1, dim(range space of L) = $\dim(\mathcal{V})$. So, dimensions of input and output vector spaces are equal implying that the matrix representation of L will be a square matrix.

(b) We now prove that L^{-1} is a linear transformation. Let $\mathbf{w}_1, \mathbf{w}_2$in $\mathcal{W}$, where $\mathsf{L}(\mathbf{v}_1) = \mathbf{w}_1$ and $\mathsf{L}(\mathbf{v}_2) = \mathbf{w}_2$, for $\mathbf{v}_1, \mathbf{v}_2$ in $\mathcal{V}$. Then, since $\mathsf{L}(a\mathbf{v}_1 + b\mathbf{v}_2) = a\mathsf{L}(\mathbf{v}_1) + b\mathsf{L}(\mathbf{v}_2) = a\mathbf{w}_1 + b\mathbf{w}_2$ for some a, b real numbers, we have, $\mathsf{L}^{-1}(a\mathbf{w}_1 + b\mathbf{w}_2) = a\mathbf{v}_1 + b\mathbf{v}_2 = a\mathsf{L}^{-1}(\mathbf{w}_1 + b\mathsf{L}^{-1}(\mathbf{w}_2)$, which implies that L^{-1} is a linear transformation. Now, say, $(\mathsf{L}^{-1})^{-1}(\mathbf{v}) = \mathbf{w}$, for some $\mathbf{v} \in \mathcal{V}$ and for some $\mathbf{w} \in \mathcal{W}$. It implies $\mathsf{L}^{-1}(\mathbf{w}) = \mathbf{v} \Rightarrow \mathsf{L}(v) = \mathbf{w}$. So, $(\mathsf{L}^{-1})^{-1} = \mathsf{L}$.

8. Consider the problem, $A\mathbf{x} = \mathbf{b}$, with $\mathbf{b} = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 14 \end{bmatrix}$. The set of all solutions is given by $\{\mathbf{x} \mid \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $c, d \in \mathbb{R}\}$

(a) Find the size of the matrix A.

(b) Find the dimension of all the four fundamental spaces of A.

(c) Find the matrix A.

Solution:

(a) Here $\mathbf{b} \in \mathbb{R}^4$, hence A has 4 rows. Also $\mathbf{x} \in \mathbb{R}^3$, hence A has 3 columns. The size of A is 4×3.

(b)

- Solution set is expressed as the shifted null space, where a particular solution is added to shift the null space. Hence it can seen from the solution set that the dimension of nullspace is 2.
- Dim(Rowspace(A)) = 3 - Dim(Nullspace(A)) = 1.
- Dim((Columnspace(A))) = Dim(Rowspace(A)) = 1
- Dim(Leftnullspace(A)) = 4 - Dim(Columnspace(A)) = 3

(c)

- Let $\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}$ be the columns of A.
- $\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{b}$.

  $\implies -2\mathbf{c_3} = \mathbf{b}$
  $\implies \mathbf{c_3} = \mathbf{b}/-2$
  $\implies \mathbf{c_3} = \begin{bmatrix} -2 \\ -3 \\ -5 \\ -7 \end{bmatrix}$
- $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is in nullspace of A.

  $\implies \mathbf{c_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is in nullspace of A.
  $\implies \mathbf{c_1} + \mathbf{c_3} = 0.$
  $\implies \mathbf{c_1} = -\mathbf{c_3}$
  $\implies c_1 = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}$
- Therefore, $A = \begin{bmatrix} 2 & 0 & -2 \\ 3 & 0 & -3 \\ 5 & 0 & -5 \\ 7 & 0 & -7 \end{bmatrix}$.

9. Matrix P is called a projector if $P^2 = P$. Suppose $\mathbf{v}$ is an n-length vector, with $\mathbf{v} \neq \mathbf{0}$ and $A = \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}$.
   (a) Prove that A is a projector.
   (b) Let I be an $n \times n$ identity matrix. Are $I - A$, $I + A$ and $A(A^TA)^{-1}A^T$ projectors? Prove your answers. Assume A^TA is invertible.
   (c) Let $\mathbf{v}_1 \neq \mathbf{0}$ be another n-length vector and $\mathbf{v}_2 = (I - A)\mathbf{v}_1$. Compute $\mathbf{v}^T\mathbf{v}_2$. What can you say about vectors $\mathbf{v}$ and $\mathbf{v}_2$?

Solution:

(a) We have the following:

$$A^2 = \left(\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\right)\left(\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\right) = \frac{\mathbf{v}(\mathbf{v}^T\mathbf{v})\mathbf{v}^T}{(\mathbf{v}^T\mathbf{v})^2} = \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} = A.$$

Hence, A is a projector.

(b) Now, $(I-A)^2 = I^2 + A^2 - IA - AI = I + A - 2A = I - A$. Thus, $I - A$ is a projector. Here, we used the fact that A is a projector. Then, $(I+A)^2 = I^2 + A^2 + IA + AI = I + 3A \neq I + A$ in general. So, $I + A$ is not a projector. Finally,

$$\begin{aligned}(A(A^TA)^{-1}A^T)^2 &= \left(A(A^TA)^{-1}A^T\right)\left(A(A^TA)^{-1}A^T\right) \\ &= A(A^TA)^{-1}\left(A^TA\right)(A^TA)^{-1}A^T = A(A^TA)^{-1}A^T.\end{aligned}$$

Hence, it is a projector.

(c) Here, we have,

$$\begin{aligned}\mathbf{v}^T\mathbf{v}_2 = \mathbf{v}^T(I-A)\mathbf{v}_1 &= \mathbf{v}^T\left(\mathbf{v}_1 - \frac{\mathbf{v}\mathbf{v}^T\mathbf{v}_1}{\mathbf{v}^T\mathbf{v}}\right) \\ &= \mathbf{v}^T\mathbf{v}_1 - \frac{(\mathbf{v}^T\mathbf{v})(\mathbf{v}^T\mathbf{v}_1)}{\mathbf{v}^T\mathbf{v}} = \mathbf{v}^T\mathbf{v}_1 - \mathbf{v}^T\mathbf{v}_1 = 0.\end{aligned}$$

Hence, vectors $\mathbf{v}$ and $\mathbf{v}_2$ are perpendicular to each other.

Matlab Section (Optional)

Useful Matlab functions: `dftmtx`(N) $\rightarrow$ Generates $N \times N$ DFT matrix, `fft`$(\mathbf{x})$ $\rightarrow$ Generates the DFT of a vector $\mathbf{x}$.

1. Computing N-point DFT of a N-length sequence $\mathbf{x}$ is a linear transformation. Assuming $N = 4$, compute the matrix representation of this linear transformation using the standard basis (i.e. by giving one basis vector after the other to the `fft` command). Verify the obtained matrix with that generated using *dftmtx* command.

2. (a) Plot the point $(3, 0)$ in Matlab.
   (b) Generate a matrix that reflects $(3, 0)$ about the $x = y$ line and plot the resultant point in the same figure obtained in (a).
   (c) Compute the matrix that can project $(3, 0)$ onto the line $x = y$ and plot the resultant point in the same figure.
   (d) Evaluate the matrix which rotates the vector $(3, 0)$ by 60^0 clockwise and plot the final obtained vector too.

Code to visualize second problem:

```
%% Program to plot vector 'u' and its transformation vector 'v=Au'
%% 'u' is of size 2X1 and 'A' of size 2X2.
```

```
%% 'u','Ax' vector can be represented with (x,y) coordinates, in 2D-plane
clc;close all;clear all;
u = [1;2]; % Initalizing vector 'u'
A = [1 -1;-1 0]; % Initalizing matrix 'A'
v = A*u; % v is the transformed vector
figure(1);plot(u(1),u(2),'bs','MarkerSize',20, 'MarkerEdgeColor','blue',...
    'MarkerFaceColor',[0 0 1]);hold on;
plot(v(1),v(2),'rs','MarkerSize',20, 'MarkerEdgeColor','red',...
    'MarkerFaceColor',[1 0 0]);grid on;grid minor;
plot(0,0,'bo','MarkerSize',10, 'MarkerEdgeColor','black',...
    'MarkerFaceColor',[0 0 0]);
legend('vector "u"','Transformed vector "v"','Origin');
minn = min(min([u(1) v(1) u(2) v(2)])-2,-1);
maxx = max(max([u(1) v(1) u(2) v(2)])+2,1);
xlim([minn,maxx]);
ylim([minn,maxx]);
set(gca,'FontSize',30);
xlabel('x ---->','fontweight','bold','fontsize',30);
ylabel('y ---->','fontweight','bold','fontsize',30);
hTitle = title('Plot of points x, Ax');
set(hTitle,'FontSize',30); axis equal;
```