EE5120 Linear Algebra: Tutorial 2, July-Dec 2018, Dr. Uday Khankhoje, EE IIT Madras
Covers Ch 2.1,2.2,2.3 of GS

1. Is the product of lower triangular matrices always lower triangular?
2. Which of the following are sub-spaces of $\mathbb{R}^3$? Justify your answer.
   (a) $\mathcal{V}_1 = \{(a_1, a_2, a_3)\Big| a_1 + a_2 + a_3 = 1\}$.
   (b) $\mathcal{V}_2 = \{(b_1, b_2, b_3)\Big| b_2 = b_3, b_1 = 2b_2\}$.
   (c) $\mathcal{V}_3 = \{(c_1, c_2, c_3)\Big| c_1 + 2c_2 + 3c_3 = 0\}$.
3. Let $\mathcal{W}$ be the set of all 2×2 matrices
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
   such that $A\mathbf{z} = 0$, where $\mathbf{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Is $\mathcal{W}$ a subspace of $\mathbb{M}_{22}$, where $\mathbb{M}_{22}$ is the vector space of all 2×2 real valued matrices? Explain.
4. Suppose $\mathcal{V}$ is a vector space. Let $\mathcal{W}_1, \mathcal{W}_2 \subset \mathcal{V}$ be sub-spaces. Which of the following sets are sub-spaces? If a set is a sub-space, prove it. Else, provide a counter-example and state under what circumstance, it can be a sub-space.
   (a) $\mathcal{W}_1 \cap \mathcal{W}_2$.
   (b) $\mathcal{W}_1 \cup \mathcal{W}_2$.
   (c) $\mathcal{W}_3 = \{\mathbf{v}\Big|\mathbf{v}^T\mathbf{u} = 0, \forall \mathbf{u} \in \mathcal{W}_1\}$.
   (d) $\mathcal{W} = \{\mathbf{w}\Big|\exists \mathbf{w}_1 \in \mathcal{W}_1, \mathbf{w}_2 \in \mathcal{W}_2 \text{ satisfying } \mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2\}$.
5. (a) The span of the following set of vectors is a sub-space of _____ dimensional real space. Fill in the blank.
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$
   (b) What is the dimension of the sub-space spanned by the vectors given in (a)?
   (c) Following (a) and (b), Check whether the following statement is true.
   Statement : If $\mathcal{S}$ is an m-dimensional vector space and $\mathcal{S} \subseteq \mathbb{R}^n$, n is always equal to m.
   If true, justify/prove it. If false, specify the possible values that n can take.
   (d) Find a set of vector(s) that yields 0 on taking inner product with any of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Comment whether the above computed set of vectors form a vector space. If it does, what is it's dimension? Compare this dimension with those obtained in (a) and (b).
6. Solve the following:
   (a) Reduce these matrices A and B to their ordinary echelon forms U:
$$(i) A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}, \quad (ii) B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$$
   Find the special solution for each free variable and describe every solution to $Ax = 0$ and $Bx = 0$. Reduce the echelon forms U to R, and draw a box around the identity matrix in the pivot rows and pivot columns.

(b) Find the column space and nullspace of A and the solution to $Ax = b$:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

7. R denotes the row-reduced echelon form of a 5x3 matrix A. R has three non-zero pivots.
   (a) Find the set of vector(s) that solve $R\mathbf{x} = \mathbf{0}$.
   (b) The matrix B is defined as $\begin{bmatrix} \mathrm{R} \\ 2\mathrm{R} \end{bmatrix}$. Find the rank of B.
   (c) The matrix C is defined as $\begin{bmatrix} \mathrm{R} & \mathrm{R} \\ \mathrm{R} & 0 \end{bmatrix}$. Find the rank of C.

8. Prove the following:
   (a) $Rank(AB) \leq Rank(A)$.
   (b) Suppose A and B are n by n matrices, and $AB = I$. Prove from $rank(AB) \leq rank(A)$ that the rank of A is n. So, A is invertible and B must be its inverse. Therefore, $BA = I$.

9. (a) Which of the following sets of vectors are linearly independent? Justify your answer.
   - $\mathcal{S}_1 = \{(0, 0, 0, 0)\}$.
   - $\mathcal{S}_2 = \{(1, 1, 1)\}$.
   - $\mathcal{S}_3 = \{(1, -1, 0), (0, 0, 1), (1, 1, 0)\}$.

   (b) Suppose $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a set of finite number of m-length vectors ($m \geq n$). If $\mathbf{v}_j \neq \mathbf{0}$, $\forall j = 1, ..., n$ and $\mathbf{v}_i^T \mathbf{v}_k = 0$, $\forall i \neq k$ and $i, k = 1, ..., n$, then prove that $\mathcal{S}$ contains linearly independent vectors.

   (c) Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be sets containing finite number of vectors such that $\mathcal{S}_1 \subset \mathcal{S}_2$. Which of the following statements is/are True? Justify your answer. Prove the statement if it is true, else given a specific counter-example if the statement is false.
   - If $\mathcal{S}_2$ is linearly dependent, then so is $\mathcal{S}_1$.
   - If $\mathcal{S}_2$ is linearly independent, then so it $\mathcal{S}_1$.

10. (a) Find a basis for the given sub-spaces of $\mathbb{R}^3$ and $\mathbb{R}^4$.
    (i) All vectors of the form, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where a=0.
    (ii) All vectors of the form, $\begin{bmatrix} a+c \\ a-b \\ b+c \\ -a+b \end{bmatrix}$.
    (iii) All vectors of the form, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a - b + 5c = 0$.

    (b) Let $\mathcal{V}$ be the vector space of 2×2 matrices, and $\mathcal{W}$ be the sub-space of symmetric matrices. Show that $\dim(\mathcal{W}) = 3$, by finding a basis of $\mathcal{W}$.

Matlab Section (Optional)

Useful Matlab Functions:
Reduced row echelon form: `rref`
Rank of the matrix: `rank`
Null space: `null`