EE5120 Linear Algebra: Tutorial 8, July-Dec 2017-18

Covers sec 6.1,6.2 (exclude law of inertia and generalized eigenvalue problem),6.3 of GS

1. Compute the SVD of $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

> **Solution:**
>
> Let SVD of A be $U\lambda V^T$. We now need to find U, Λ and V. Note that $AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $A^TA = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. Eigenvalues of AA^T are $\lambda_1 = \lambda_2 = 2 \Rightarrow$ the singular values are $\sigma_1 = \sigma_2 = \sqrt{2}$. Orthonormal eigenvectors for AA^T can be given by $\mathbf{u}_1 = [1\,0]^T$ and $\mathbf{u}_2 = [0\,1]^T$. Thus, $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Two orthonormal eigenvectors for A^TA can be found immediately as $\mathbf{v}_1 = \frac{1}{\sqrt{2}}[1\,0\,1\,0]^T$ and $\mathbf{v}_2 = \frac{1}{\sqrt{2}}[0\,1\,0\,1]^T$. We need to find two more vectors $\mathbf{v}_3$ and $\mathbf{v}_4$ orthogonal to $\{\mathbf{v}_i\}_{i=1}^2$ such that $\{\mathbf{v}_i\}_{i=1}^4$ will form orthonormal basis for $\mathbb{R}^4$. Then, $\mathbf{v}_3 = \frac{1}{\sqrt{2}}[1\,0\,-1\,0]^T$ and $\mathbf{v}_4 = \frac{1}{\sqrt{2}}[0\,1\,0\,-1]^T$. Thus,
>
> $$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

2. (a) Prove that a symmetric matrix A is positive definite if and only if there exists a matrix B with independent columns such that $A = B^TB$.

   (b) If A is written as its eigenvalue decomposition, what will B be ?

Hint: Substitute A in x^TAx and solve.

> **Solution:** Part (a): For positive definiteness, $x^TAx > 0$ should satisfy for all nonzero vectors x. Using $A = B^TB$, we get $x^TAx = x^TB^TBx = (Bx)^T(Bx) = \|Bx\|^2$. This squared length is positive (unless $x = 0$), because B has independent columns. (If x is nonzero then Bx is nonzero). Thus B^TB is positive definite.
>
> Part (b): Eigenvalue decomposition of A is $A = Q\Lambda Q^T = (Q\sqrt{\Lambda})(\sqrt{\Lambda}Q^T)$, so $B = \sqrt{\Lambda}Q^T$

3. Suppose A is a 2 by 2 symmetric matrix with unit eigenvectors u_1 and u_2. If its eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$, what are U, Σ and V^T ?

Hint: Use property of symmetric matrices to find singular values.

> **Solution:** Since $A = A^T$, $A^TA = A^2 = V\Sigma U^TU\Sigma V^T = V\Sigma^2V^T = (V\Sigma V^T)^2$. Thus, $A = U\Sigma V^T = V\Sigma V^T$. This implies that matrices U and V will satisfy $U = V$ and the columns of U (or V) will consist u_1 and u_2. Since, singular values are always non-negative, we have $\sigma_1 = \lambda_1 = 3$ and $\sigma_2 = -\lambda_2 = 2$. So, $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

4. For what range of a and b are the matrices **A**,**B** positive definite

$$\mathbf{A} = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{bmatrix}$$

*Hint:*Use definitions.

Solution: **A** is Positive Definite when $a > 2$, whereas **B** is never Positive Definite

5. Let **A** and **B** be real square symmetric Positive semi-definite matrices. Is **AB**+**BA** positive semi-definite always

*Hint:*Counter examples?.

Solution: consider following PSD matrices

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

for $\mathbf{x} = (0,1)^t$, $\mathbf{x}^t(\mathbf{AB}+\mathbf{BA})\mathbf{x} = -6 < 0$

6. Give a quick reason why each of these statements is true:
   (a) Every positive definite matrix is invertible.
   (b) The only positive definite projection matrix is $P = I$.
   (c) A diagonal matrix with positive diagonal entries is positive definite.
   (d) A symmetric matrix with a positive determinant might not be positive definite

Solution:
(a) The determinant is positive (not zero) as all eigenvalues are positive.
(b) All projection matrices except I are singular (non-invertible).
(c) The diagonal entries of a diagonal matrix are its eigenvalues.
(d) $A = -I$ has determinant equal to 1 when n is even, but it is not a positive-definite matrix.

7. Suppose $\mathbf{u_1}, ..., \mathbf{u_n}$ and $\mathbf{v_1}, ..., \mathbf{v_n}$ are orthonormal bases for R_n. Construct the matrix A that transforms each $\mathbf{v_j}$ into $\mathbf{u_j}$ to give $A\mathbf{v_1} = \mathbf{u_1}, ..., A\mathbf{v_n} = \mathbf{u_n}$.

*Hint:*Write the equations in matrix form to find expression for A.

Solution: Let U be the matrix whose columns are $\mathbf{u_1}, ..., \mathbf{u_n}$ and let V be the matrix whose columns are $\mathbf{v_1}, ..., \mathbf{v_n}$. Then, the condition $A\mathbf{v_1} = \mathbf{u_1}, ..., A\mathbf{v_n} = \mathbf{u_n}$ can be written as

$$AV = U.$$

Hence

$$A = UV^{-1} = UV^T.$$

Now, A is an orthogonal matrix because

$$A^T A = (UV^T)^T UV^T = V(U^T U)V^T = VV^T = I.$$

Note also that $A = UIV^T$ is the SVD for A, where the singular value matrix $\Sigma = I$.

8. Let A be an $n \times n$ hermitian matrix with n distinct eigenvalues. Prove that for any n length vector $\mathbf{x}$, $\lambda_{\min}||\mathbf{x}||^2 \leq \mathbf{x}^H A\mathbf{x} \leq \lambda_{\max}||\mathbf{x}||^2$, where $||\mathbf{x}||^2 = \mathbf{x}^H\mathbf{x}$, $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of A.

Hint: Use eigen decomposition of A and its properties (refer to Q3 of prev. tutorial).

Solution:

Since A is a hermitian matrix with all its eigenvalues to be distinct, the eigenvalue decomposition of A can be written as, $A = U\Lambda U^H$, where U is the eigenvector matrix which is unitary and Λ is a diagonal matrix containing eigenvalues of A as its diagonal entries (refer to Q3 of prev. tutorial). Let $\mathbf{y} = U^H\mathbf{x}$ and $\mathbf{y}(k)$ denote k^{th} element of $\mathbf{y}$. Then, we have the following:

$$\mathbf{x}^H A\mathbf{x} = \mathbf{x}^H U\Lambda U^H\mathbf{x} = \mathbf{y}^H\Lambda\mathbf{y} = \sum_{k=1}^{n} \lambda_k|\mathbf{y}(k)|^2,$$

where λ_k is the k^{th} diagonal entry in Λ. Now,

$$\begin{aligned}\sum_{k=1}^{n} \lambda_k|\mathbf{y}(k)|^2 \leq \sum_{k=1}^{n} \lambda_{\max}|\mathbf{y}(k)|^2 &= \lambda_{\max}\sum_{k=1}^{n} |\mathbf{y}(k)|^2 \\ &= \lambda_{\max}||\mathbf{y}||^2 = \lambda_{\max}||U^H\mathbf{x}||^2 = \lambda_{\max}||\mathbf{x}||^2.\end{aligned}$$

Similarly, we can get, $\sum_{k=1}^{n} \lambda_k|\mathbf{y}(k)|^2 \geq \lambda_{\min}||\mathbf{x}||^2$. Hence, proved the result.

9. The graph of $F_1(x,y) = x^2 + y^2$ is a bowl opening upward. The graph of $F_3(x,y) = x^2 - y^2$ is a saddle. The graph of $F_3(x,y) = -x^2 - y^2$ is a bowl opening downward. What is a test on $F(x,y)$ for having maxima, minima or saddle point at $(0,0)$?

Hint: Use second derivative matrix of the function.

Solution: First derivatives of the function F(x,y) should be zero at $(0,0)$. It is satisfied for all the three functions at $(0,0)$. So, all are having a stationary point at $(0,0)$.

Second derivative matrices for $x^2+y^2, x^2-y^2, and\ -x^2-y^2$ are given below:

$$F_1(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, F_2(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} and F_3(x,y) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

At minima, the second derivative matrix will have all positive eigenvalues(positive definite matrix). So, $F_1(x,y)$ is having a minima.
At saddle, the second derivative matrix will have atleast one positve and one negative eigenvalue. So, $F_2(x,y)$ is a saddle.
At maxima, the second derivative matrix will have all negative eigenvalues(negative definite matrix). So, $F_3(x,y)$ is having a maxima.

10. (a) If A changes to $4A$, what is the change in the SVD?
    (b) What is the SVD for A^T and for A^{-1}?
    (c) Why doesn't the SVD for $A+I$ just use $\Sigma + I$?

Hint: Calculate SVD of (A+I) in terms of SVD of A.

Solution: Let $A = U\Sigma V^T$

(a) $4A = 4(U\Sigma V^T) = U(4\Sigma)V^T$.

(b) $A^T = (U\Sigma V^T)^T = V^{T^T}\Sigma^T U^T = V\Sigma^T U^T$.
$A^{-1} = (U\Sigma V^T)^{-1} = V^{T^{-1}}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T$.
Where Σ^{-1} is a Diagonal matrix of size Σ^T and Diagonal elements as $1/\sigma_{ii}$.

If a singularvalue is zero, then we need to fix the corresponding singularvalue of A^{-1} to zero. But, We will have only a pseudo-inverse.

For non-square matrices will have leftside and rightside inverse.

(c)

$$\begin{aligned} (A+I)(A+I)^T &= AA^T + AI^T + IA^T + II^T \\ &= U\Sigma\Sigma^T U^T + U\Sigma V^T I^T + I(V\Sigma^T U^T) + UII^T U^T \end{aligned} \tag{1}$$

If $A + I = U(\Sigma + I)V^T$, where A and I are of size $M \times N$.

$$\begin{aligned} \Rightarrow (A+I)(A+I)^T &= U(\Sigma+I)(\Sigma+I)^T U^T \\ &= U\Sigma\Sigma^T U^T + U\Sigma I^T U^T + UI\Sigma^T U^T + UII^T U^T \end{aligned} \tag{2}$$

$\Rightarrow$ Eq (1) $\neq$ (2).
Even if A is square, Eq(1) $\neq$ Eq(2) as $U \neq V$.