EE5120 Linear Algebra: Tutorial Test 6, 27.10.17B

Give your answers in the space provided. No calculators or smartphones allowed.

Roll: No:________________ NAME:________________________ Time: 20 mins

$\boxed{8}$ 1. In class we discussed the spectral theorem which states that every real, symmetric matrix A admits a decomposition of the form $A = Q\Lambda Q^T$ (where Λ is a diagonal matrix of eigenvalues and Q is a orthonormal matrix of n eigenvectors). Prove the above theorem by starting with the result of the Schur decomposition theorem (i.e. state and use the decomposition, but don't prove it).
Note: We proved this in class when the eigenvalues were distinct. Now a general proof is asked. Finally comment on whether or not the repetition of eigenvalues matters in this proof. Every step must be properly justified to get credit.

> **Solution:** Schur decomposition says that any matrix A can be expressed in terms of a unitary matrix U and a triangular matrix R as $A = URU^{-1}$ [1]
> Now, A is real, symmetric, so all the eigenvalues and vectors are real [1].
> We get $A = QRQ^{-1}$ where Q is orthogonal (note $Q^{-1} = Q^T$). Next, $A^T = A$, but $A^T = QR^TQ^T$ implying that $R^T = R$, which is possibly only if R is a diagonal matrix, Λ. [2]
> Now, right multiply A by Q to get $AQ = Q\Lambda$. This is expanded as:
> $A[q_1\, q_2 \ldots q_n] = [q_1\, q_2 \ldots q_n] \begin{bmatrix} \lambda_1 & 0 & \ldots \\ 0 & \lambda_2 & \cdots \\ \vdots & 0 & \ddots \end{bmatrix}$ Thus, it is easily seen that q_is are eigenvectors $Aq_i = \lambda_i q_i$. [2]
> So regardless of whether or not an eigenvalue is repeated, we have a full set of eigenvectors that form Q and we have shown that $A = Q\Lambda Q^T$ for real symmetric A. [2]

$\boxed{2}$ 2. Given two vectors x, y in $\mathbb{C}^n$, prove that a unitary transformation does not alter the distance between these vectors.

> **Solution:** Let $d = \|x - y\|$. Then the distance between Ux and Uy after transformation becomes $d' = \|Ux - Uy\| = \|U(x - y)\| = \sqrt{(x - y)^H U^H U(x - y)} = d$.
> (Deduct 0.5 if transpose used instead of hermitian)