EE5120 Linear Algebra: Tutorial 6, July-Dec 2017-18
Covers sec 4.2, 5.1, 5.2 of GS

1. State True or False with proper explanation:
   (a) All vectors are eigenvectors of the Identity matrix.
   (b) Any matrix can be diagonalized.
   (c) Eigenvalues must be nonzero scalars.
   (d) A and B are said to be *Similar* matrices if there exists an invertible matrix P such that $P^{-1}AP = B$. A and B always have the same eigenvalues.
   (e) The sum of two eigenvectors of an operator $\mathbf{T}$ is always an eigenvector of $\mathbf{T}$.

2. Let $\mathbf{T}$ be the linear operator on n x n real matrices defined by $\mathbf{T}(A) = A^t$. Show that ± 1 are the only eigenvalues of $\mathbf{T}$. Describe the eigenvectors corresponding to each eigenvalue of $\mathbf{T}$.

   Hint: Write the Eigenvalue equation as $T(A) = A^t = \lambda A$ and proceed.

3. Prove that the geometric multiplicity of an eigenvalue, $\mu_A(\lambda_i)$, can not exceed its algebraic multiplicity, $\gamma_A(\lambda_i)$. Thus, from here conclude (and prove that) $1 \leq \gamma_A(\lambda_i) \leq \mu_A(\lambda_i) \leq n$

4. Consider the following $N \times N$ matrix:

$$\mathbf{A} = \begin{bmatrix} x & -x & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ x & x & -x & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & x & x & -x & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & x & x & -x & 0 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots & x & x & -x \\ 0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots & 0 & x & x \end{bmatrix}$$

   This implies for $N = 1,2,3$, matrix $\mathbf{A}$ looks like,

$$\begin{bmatrix} x \end{bmatrix} \quad \begin{bmatrix} x & -x \\ x & x \end{bmatrix} \quad \begin{bmatrix} x & -x & 0 \\ x & x & -x \\ 0 & x & x \end{bmatrix}$$

   Show that the determinant of $\mathbf{A}$ is $(F_{N-1} + F_{N-2})x^N$, where $F_1 = 1$, $F_2 = 2$ and $F_N = F_{N-1} + F_{N-2}$.

   Hint: Use Mathematical Induction.

5. Let $p(\lambda) = \prod_{i=1}^{n} (\lambda_i - \lambda)$ be the characteristic polynomial of the $n \times n$ matrix $\mathbf{A}$. Derive the characteristic polynomial of $\mathbf{A}^2 - \mathbf{I}$, where $\mathbf{I}$ is an identity matrix of appropriate dimension.

   Hint: Use properties of eigen values and definition of a characteristic polynomial.

6. Prove that a linear transformation $\mathbf{T}$ on a finite dimensional vector space is inverible iff zero is not an eigen value of $\mathbf{T}$

   Hint: Use properties of eigen values.

7. (a) What is wrong with this proof that projection matrices have $det\, P = 1$?

$$P = A(A^T A)^{-1} A^T \qquad \text{so} \qquad |P| = |A| \frac{1}{|A^T||A|} |A^T| = 1$$

Hint: Invertibility.

(b) Suppose the 4by4 matrix M has four equal rows all containing a, b, c, d. We know that $det(M) = 0$. Find the $det(I + M)$ by any method?

$$det(I+M) = \begin{bmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{bmatrix}$$

Hint: Use properties of determinants.

8. Find the eigenvalues and eigenvectors for both of these Markov matrices A and A^∞. Expain why A^{100} is close to A^∞:

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \qquad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$$

Hint: Use diagonalization.

9. When $a + b = c + d$, show that (1,1) is an eigenvector and find both eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Hint: Use definition of eigen vector, $Ax = \lambda x$ and substitute given vector for x.

10. EXTRA: Find $u(t)$ that satisfies the differential equation $du/dt = Pu$, when P is a projection:

$$\frac{du}{dt} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} u \quad \text{with} \quad u(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Here $u(t)$ is a vector of time-varying functions, i.e., we can write $u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$. You will find that a part of u increases exponentially while another part stays constant.

Hint: Find eigen values and eigen vectors of P and substitute given initial condition.