EE5120 Linear Algebra: Tutorial 6, July-Dec 2017-18

Covers sec 4.2, 5.1, 5.2 of GS

1. State True or False with proper explanation:
   (a) All vectors are eigenvectors of the Identity matrix.
   (b) Any matrix can be diagonalized.
   (c) Eigenvalues must be nonzero scalars.
   (d) A and B are said to be *Similar* matrices if there exists an invertible matrix P such that $P^{-1}AP = B$. A and B always have the same eigenvalues.
   (e) The sum of two eigenvectors of an operator $\mathbf{T}$ is always an eigenvector of $\mathbf{T}$.

Solution:

(a) **True**. We know, $S^{-1}AS = \Lambda$. If $A = I$, $S^{-1}IS$ is always diagonal (Λ is just I). The only requirement is that S should be invertible.

(b) **False**. Any matrix with distinct eigenvalues can be diagonalized.

(c) **False**. They can be zero as well. But, eigenvectors have to be nonzero. Having zero eigenvalue implies that the matrix is non-invertible.

(d) **True**. If A and B are similar, there is some invertible matrix P such that $P^{-1}AP = B$. Thus, $P^{-1}A = BP^{-1}$ or $AP = PB$.
If $Av = \lambda v$, we have $B(P^{-1}v) = \lambda P^{-1}v$. Similarly, if $Bv = \lambda v$, then we have $A(Pv) = \lambda Pv$. Thus both have same eigenvalues λ.

(e) **False**. For example, vectors $(1, -1)^t$ and $(0, 1)^t$ are eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

But the sum of them $(1, 0)^t$ is not an eigenvector of the same matrix.

2. Let $\mathbf{T}$ be the linear operator on n x n real matrices defined by $\mathbf{T}(A) = A^t$. Show that ± 1 are the only eigenvalues of $\mathbf{T}$. Describe the eigenvectors corresponding to each eigenvalue of $\mathbf{T}$.

Hint: Write the Eigenvalue equation as $T(A) = A^t = \lambda A$ and proceed.

Solution: If $T(A) = A^t = \lambda A$ for some λ and some nonzero matrix A, say $A_{ij} \neq 0$, we have

$$A_{ij} = \lambda A_{ji}$$

and

$$A_{ji} = \lambda A_{ij}$$

and so

$$A_{ij} = \lambda^2 A_{ij}$$

This means that λ can be only 1 or -1. And these two values are eigenvalues due to the existence of symmetric and skew-symmetric matrices.

The set of nonzero symmetric matrices are the eigenvectors corresponding to eigenvalue 1, while the set of nonzero skew-symmetric matrices are the eigenvectors corresponding to eigenvalue -1.

3. Prove that the geometric multiplicity of an eigenvalue, $\gamma_A(\lambda_i)$, can not exceed its algebraic multiplicity, $\mu_A(\lambda_i)$. Thus, from here conclude (and prove that) $1 \leq \gamma_A(\lambda_i) \leq \mu_A(\lambda_i) \leq n$

Solution: See `http://www.ee.iitm.ac.in/uday/2017b-EE5120/multiplicity.pdf`

4. Consider the following $N \times N$ matrix:

$$\mathbf{A} = \begin{bmatrix} x & -x & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ x & x & -x & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & x & x & -x & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & x & x & -x & 0 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & ... & ... & ... & x & x & -x \\ 0 & 0 & 0 & ... & ... & ... & ... & 0 & x & x \end{bmatrix}$$

This implies for $N = 1, 2, 3$, matrix $\mathbf{A}$ looks like,

$$\begin{bmatrix} x \end{bmatrix} \quad \begin{bmatrix} x & -x \\ x & x \end{bmatrix} \quad \begin{bmatrix} x & -x & 0 \\ x & x & -x \\ 0 & x & x \end{bmatrix}$$

Show that the determinant of $\mathbf{A}$ is $(F_{N-1} + F_{N-2})x^N$, where $F_1 = 1$, $F_2 = 2$ and $F_N = F_{N-1} + F_{N-2}$.

Hint: Use Mathematical Induction.

Solution: (Use Mathematical Induction) It can be easily verified that the result is true for $N = 1, 2, 3$. Let us assume that it is true upto dimension $N-1 \times N-1$, where $N > 4$. If we prove that the result holds for when the matrix dimension is $N \times N$, then we are done. Carefully observe that the determinant of the $N \times N$ matrix $\mathbf{A}$ is given by,

$$\begin{aligned} x(\text{det. of } N-1 \times N-1 \text{ matrix}) - (-x)\Big(x(\text{det. of } N-2 \times N-2 \text{ matrix})\Big) \\ &= x\Big((F_{N-2} + F_{N-3})x^{N-1}\Big) + x^2\Big((F_{N-3} + F_{N-4})x^{N-2}\Big) \\ &= F_{N-1}x^N + F_{N-2}x^N. \end{aligned}$$

Hence proved.

Alt proof: Partition A_n as $A_n = \begin{bmatrix} x & -x & \dots \\ x & A_{n-1} & \\ \vdots & & \ddots \end{bmatrix}$.

OR

(Using row transformations) Using the property that subtracting a multiple of one row from another row leaves the same determinant.
Subtracting $Row2 = Row2 - Row1$, followed by $Row3 = Row3 - \frac{Row2}{2}$, and soon to matrix A to get a upper triangular matrix as shown below,

$$det(\mathbf{A}) = \begin{vmatrix} x & -x & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & \frac{2}{1}x & -x & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2}x & -x & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{3}x & -x & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{8}{5}x & -x & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \end{vmatrix}$$

It can be observed that the diagonal elements are of the following form $a_{ii} = \frac{F_{ii-1}}{F_{ii-2}}x$.

$$det(\mathbf{A}) = \begin{vmatrix} x & -x & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & \frac{F_2}{F_1}x & -x & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{F_3}{F_2}x & -x & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{F_4}{F_3}x & -x & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & ... & ... & ... & 0 & \frac{F_{N-1}}{F_{N-2}}x & -x \\ 0 & 0 & 0 & ... & ... & ... & ... & 0 & 0 & \frac{F_N}{F_{N-1}}x \end{vmatrix}$$

As the matrix is upper triangular, determinant is product of diagonal elements,
i.e. $det(\mathbf{A}) = \frac{F_1}{F_0}x \times \frac{F_1}{F_2}x....\frac{F_{N-1}}{F_{N-2}}x \times \frac{F_N}{F_{N-1}}x = F_N x^N = (F_{N-1} + F_{N-2})x^N$

5. Let $p(\lambda) = \prod_{i=1}^{n}(\lambda_i - \lambda)$ be the characteristic polynomial of the $n \times n$ matrix $\mathbf{A}$. Derive the characteristic polynomial of $\mathbf{A}^2 - \mathbf{I}$, where $\mathbf{I}$ is an identity matrix of appropriate dimension.

Hint: Use properties of eigen values and definition of a characteristic polynomial.

Solution:

Let Λ be such that,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & ... & 0 \\ 0 & \lambda_2 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \lambda_n \end{bmatrix}$$

Then, there exists an invertible matrix $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}\Lambda\mathbf{B}^{-1}$. Further, we get $\mathbf{A}^2 =$

$\mathbf{B}\Lambda\mathbf{B}^{-1}\mathbf{B}\Lambda\mathbf{B}^{-1} = \mathbf{B}\Lambda^2\mathbf{B}^{-1}$. Now, the characteristic polynomial of $\mathbf{A}^2 - \mathbf{I}$ is given by,

$$\begin{aligned}
\det\Big((\mathbf{A}^2 - \mathbf{I}) - \lambda\mathbf{I}\Big) &= \det(\mathbf{B}\Lambda^2\mathbf{B}^{-1} - (\lambda+1)\mathbf{I}) = \det(\mathbf{B}\Lambda^2\mathbf{B}^{-1} - (\lambda+1)\mathbf{B}\mathbf{B}^{-1}) \\
&= \det\Big(\mathbf{B}(\Lambda^2 - (\lambda+1)\mathbf{I})\mathbf{B}^{-1}\Big) = \det(\mathbf{B})\det(\Lambda^2 - (\lambda+1)\mathbf{I})\det(\mathbf{B}^{-1}) \\
&= \det(\Lambda^2 - (\lambda+1)\mathbf{I}).
\end{aligned}$$

Note that $\Lambda^2 - (\lambda+1)\mathbf{I}$ is a diagonal matrix. Hence, the characteristic polynomial is $\prod_{i=1}^{n}(\lambda_i^2 - (\lambda+1))$.

6. Prove that a linear transformation $\mathbf{T}$ on a finite dimensional vector space is inverible iff zero is not an eigen value of $\mathbf{T}$

   Hint: Use properties of eigen values.

Solution: $\mathbf{T}$ is invertible iff det($\mathbf{T}$) $\neq 0$. det($\mathbf{T}$)=product of eigen values. det($\mathbf{T}$) $\neq 0 \Rightarrow$ eigen values are non zero

7. (a) What is wrong with this proof that projection matrices have $det\ P = 1$?

$$P = A(A^TA)^{-1}A^T \qquad \text{so} \qquad |P| = |A|\frac{1}{|A^T||A|}|A^T| = 1$$

   Hint: Invertibility.

   (b) Suppose the 4by4 matrix M has four equal rows all containing a, b, c, d. We know that $det(M) = 0$. Find the $det(I + M)$ by any method?

$$det(I+M) = \begin{bmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{bmatrix}$$

   Hint: Use properties of determinants.

Solution:

(a) The proof is valid only if A is square and invertible, which is not the case every-time. If A is not invertible, then $(A^TA)^{-1} \neq A^{-1}A^{T^{-1}} \Rightarrow |(A^TA)^{-1}| \neq \frac{1}{|A||A^T|}$.

(b) Using the property that subtracting a multiple of one row from another row leaves the same determinant.

$$det(I+M) = \begin{bmatrix} 1+a & b & c & d \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Using the property that subtracting a multiple of one column from another column leaves the same determinant.

$$det(I+M) = \begin{bmatrix} 1+a+b+c+d & b & c & d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1+a+b+c+d$$

8. Find the eigenvalues and eigenvectors for both of these Markov matrices A and A^{∞}. Expain why A^{100} is close to A^{∞}:

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \qquad A^{\infty} = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$$

Hint: Use diagonalization.

Solution: Eigen values and eigenvectors of A are $0.4, 1$ and $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$, respectively.

Eigen values and eigenvectors of A^{∞} are $0, 1$ and $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$, respectively.

We could see that the eigenvectors are linearly independent for A. So, matrix A can be diagonalizable as shown below.

$$A = S\Lambda S^{-1}$$

where $S = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{5} \\ 1/\sqrt{2} & 2/\sqrt{5} \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 0.4 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\Rightarrow A^2 = S\Lambda S^{-1} S\Lambda S^{-1} = S\Lambda^2 S^{-1}$$

Similarly,

$$\Rightarrow A^n = S\Lambda^n S^{-1} \quad \text{and} \quad A^{\infty} = S\Lambda^{\infty} S^{-1} = S\Lambda^{\infty} S^{-1}$$

where $\Lambda^{\infty} = \begin{bmatrix} 0.4^{\infty} & 0 \\ 0 & 1^{\infty} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. So, the eigen values with magnitude less than the one, will have less significance at higher powers. $0.4^{100} = 1.6069 \times 10^{-40} \approx 0$.

9. When $a+b = c+d$, show that (1,1) is an eigenvector and find both eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Hint: Use definition of eigen vector, $Ax = \lambda x$ and substitute given vector for x.

Solution: Let $a+b = c+d = f$. Eigen vector is the solution to the equation $Ax = \lambda x$. If (1,1) is an eigen vector, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} f \\ f \end{bmatrix} = f \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which satisfies the equation $Ax = \lambda x$ with eigen value $\lambda = f = a + b = c + d$.

10. EXTRA: Find $u(t)$ that satisfies the differential equation $du/dt = Pu$, when P is a projection:

$$\frac{du}{dt} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} u \quad \text{with} \quad u(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Here $u(t)$ is a vector of time-varying functions, i.e., we can write $u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$. You will find that a part of u increases exponentially while another part stays constant.

Hint: Find eigen values and eigen vectors of P and substitute given initial condition.

Solution: Solving $det(P - \lambda I) = 0$ gives the eigen values of P as $\lambda_1 = 1$ and $\lambda_2 = 0$ (This is true for all projection matrices).
Solving $Px = \lambda x$ for each of the eigen values gives the corresponding eigen vectors as $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
In the differential equation, this produces the special solutions $u = e^{\lambda t}x$. They are the pure exponential solutions to $du/dt = Pu$. Let these be $u_1 = e^{\lambda_1 t}x_1$ and $u_2 = e^{\lambda_2 t}x_2$. Any linear combinations of u_1 and u_2 will also be solutions to the differential equation. The complete solution is given by $u(t) = c_1 e^{\lambda_1 t}x_1 + c_2 e^{\lambda_2 t}x_2$. We now find c_1 and c_2 using the initial condition $u(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, i.e.,

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

This gives $c_1 = 4$ and $c_2 = 1$. Therefore the complete solution is $u(t) = 4e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
Here, the first part of u increases exponentially while the nullspace part (corresponding to eigen value 0) remains fixed.