EE5120 Linear Algebra: Tutorial 3, July-Dec 2017-18

1. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two subsets of a vector space V such that $\mathcal{S}_1 \subset \mathcal{S}_2$. Say True/False for each of the following. If True, prove it. If False, justify it.
   (a) If $\mathcal{S}_1$ is linearly independent, then $\mathcal{S}_2$ is so.
   (b) If $\mathcal{S}_1$ is linearly dependent, then $\mathcal{S}_2$ is so.
   (c) If $\mathcal{S}_2$ is linearly independent, then $\mathcal{S}_1$ is so.
   (d) If $\mathcal{S}_2$ is linearly dependent, then $\mathcal{S}_1$ is so.

2. Let P_2 be the set of all second degree polynomials. Clearly, it is a vector space. Which of the following sets are the bases for P_2? Justify your answer.
   (a) $\{-1 - x + 2x^2, 2 + x - 2x^2, 1 - 2x + 4x^2\}$.
   (b) $\{1 + 2x + x^2, 3 + x^2, x + x^2\}$.
   (c) $\{1 - 2x - 2x^2, -2 + 3x - x^2, 1 - x + 6x^2\}$.
   (d) $\{-1 + 2x + 4x^2, 3 - 4x - 10x^2, -2 - 5x - 6x^2\}$.
   (e) $\{1 + 2x - x^2, 4 - 2x + x^2, -1 + 18x - 9x^2\}$.

3. Consider a vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$ in $\mathbb{R}^4$. It has 24 rearrangements like (x_2, x_1, x_3, x_4), (x_4, x_3, x_1, x_2), and so on. Those 24 vectors, including x itself, span a subspace S. Find specific vectors $\mathbf{x}$ so that the dimension of S is: (a) 0, (b) 1, (c) 3, (d) 4.

4. Find the basis for the following subspaces of $\mathbb{R}^5$
   (a) $\mathsf{W}_1 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 : a_1 - a_3 - a_4 = 0\}$
   (b) $\mathsf{W}_2 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 : a_2 = a_3 = a_4, a_1 + a_5 = 0\}$

5. Do the polynomials $x^3 - 2x^2 + 1$, $4x^2 - x + 3$, $3x - 2$ generate $\mathsf{P}_3(\mathbb{R})$? where $\mathsf{P}_3(\mathbb{R})$ is the set of all polynomials having degree ≤ 3

6. Prove that any rank 1 matrix has the form $\mathbf{A} = \mathbf{u}\mathbf{v}^T =$ column times row.

7. Let V be a finite dimensional vector space and let $\mathcal{S}$ be a spanning subset of V. Prove that there exists a subset of $\mathcal{S}$ that is the basis for V.

8. Prove that number of basis vectors of a vector space is unique.

9. Let $\mathbf{A}$ be an $m \times n$ matrix. Prove that the sum of dimensions of the column space and null space of $\mathbf{A}$ equals n.

10. Let $\mathbf{A}$ be an $n \times n$ invertible conjugate symmetric matrix, i.e., $\mathbf{A}^H = (\mathbf{A}^*)^T = \mathbf{A}$ ($*$ - denotes conjugation) and $\mathbf{x}$ be an $n \times 1$ vector. The following procedure guides you to find the inverse of $\mathbf{A} + \mathbf{x}\mathbf{x}^H$.
    (a) For an arbitrary $\mathbf{y}$, consider the equation,
    $$(\mathbf{A} + \mathbf{x}\mathbf{x}^H)\mathbf{z} = \mathbf{y}. \tag{1}$$
    Now finding inverse of $\mathbf{A} + \mathbf{x}\mathbf{x}^H$ is equivalent to finding a $\mathbf{B}$ such that $\mathbf{z} = \mathbf{B}\mathbf{y}$. Pre-multiply both sides of equation (1) by $\mathbf{A}^{-1}$ and obtain
    $$\mathbf{z} = \mathbf{A}^{-1}\mathbf{y} - \mathbf{A}^{-1}\mathbf{x}\mathbf{x}^H\mathbf{z}. \tag{2}$$
    (b) Pre-multiply both sides of equation (2) by $\mathbf{x}^H$ and then solve for $\mathbf{x}^H\mathbf{z}$ in terms of $\mathbf{x}, \mathbf{A}, \mathbf{y}$.
    (c) Substitute into equation (1) and manipulate to bring into the desired form $\mathbf{z} = \mathbf{B}\mathbf{y}$. Observe what $\mathbf{B}$ is.