EE5120 Linear Algebra: Tutorial 3, July-Dec 2017-18

1. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two subsets of a vector space V such that $\mathcal{S}_1 \subset \mathcal{S}_2$. Say True/False for each of the following. If True, prove it. If False, justify it.

   (a) If $\mathcal{S}_1$ is linearly independent, then $\mathcal{S}_2$ is so.

   (b) If $\mathcal{S}_1$ is linearly dependent, then $\mathcal{S}_2$ is so.

   (c) If $\mathcal{S}_2$ is linearly independent, then $\mathcal{S}_1$ is so.

   (d) If $\mathcal{S}_2$ is linearly dependent, then $\mathcal{S}_1$ is so.

Solution:

(a) False. Need not be the case always. If $\mathcal{S}_2$ contains a linear combination of elements of $\mathcal{S}_1$, then it will not be linearly independent. For example, let $\mathcal{S}_1 = \{1\}$ and $\mathcal{S}_2 = \{1,2\}$ both subsets of $\mathcal{R}$. Here $\mathcal{S}_1$ is linearly independent, while $\mathcal{S}_2$ is not.

(b) True. Since $\mathcal{S}_1$ is linearly dependent we have nite vectors $x_1, x_2, ..., x_n$ in $\mathcal{S}_1$ and so in $\mathcal{S}_2$ such that $a_1x_1 + a_2x_2 + \ + a_nx_n$=0 is a nontrivial representation. But this nontrivial representation is also a nontrivial representation of $\mathcal{S}_2$. Therefore $\mathcal{S}_2$ is also linearly dependent.

(c) True. Proof by contradiction: Assume $\mathcal{S}_1$ is linearly dependent. Then we can find a non-trivial representation for $\mathcal{S}_1$, which will also be a non-trivial representation for $\mathcal{S}_2$ and contradicts the fact that $\mathcal{S}_2$ is linearly independent.

(d) False. Need not be the case always. Refer example for (a)

2. Let P_2 be the set of all second degree polynomials. Clearly, it is a vector space. Which of the following sets are the bases for P_2? Justify your answer.

   (a) $\{-1 - x + 2x^2, 2 + x - 2x^2, 1 - 2x + 4x^2\}$.

   (b) $\{1 + 2x + x^2, 3 + x^2, x + x^2\}$.

   (c) $\{1 - 2x - 2x^2, -2 + 3x - x^2, 1 - x + 6x^2\}$.

   (d) $\{-1 + 2x + 4x^2, 3 - 4x - 10x^2, -2 - 5x - 6x^2\}$.

   (e) $\{1 + 2x - x^2, 4 - 2x + x^2, -1 + 18x - 9x^2\}$.

Solution:

Each of the given sets has three polynomials. Since dimension of P_2 is 3, it is sufficient to test linear independency for the sets to be a basis for P_2 *(Note that dimension of a vector space is the total number of basis vectors present in a basis set of that vector space)*. Test procedure is as follows: Let the polynomials in each set be $\{p_1(x), p_2(x), p_3(x)\}$. Solve $ap_1(x) + bp_2(x) + cp_3(x) = 0$ for constants a, b and c. If these constants are all zero, then the set is a linearly independent. Else, it is not. According to this, we obtain that the sets in (b), (c) and (d) are the bases for P_2.

3. Consider a vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$ in $\mathbb{R}^4$. It has 24 rearrangements like (x_2, x_1, x_3, x_4), (x_4, x_3, x_1, x_2), and so on. Those 24 vectors, including x itself, span a subspace S. Find specific vectors $\mathbf{x}$ so that the dimension of S is: (a) 0, (b) 1, (c) 3, (d) 4.

Solution:

(a) A subspace of dimension 0 in $\mathbb{R}^4$ is the origin. So the vector that spans it is (0,0,0,0).

(b) A subspace of dimension 1 implies it has only one vector in its basis. This is possible only if the vector does not change on rearrangement. One such vector is (1,1,1,1).

(c) Consider a non-zero vector (x_1, x_2, x_3, x_4) such that $x_1 + x_2 + x_3 + x_4 = 0$, i.e., its dot product with (1,1,1,1) is 0. This vector along with its permutations will span a 3-dimensional subspace that is perpendicular to the line through (1,1,1,1) and origin.

(d) Take the standard basis for $\mathbb{R}^4$, i.e., (1,0,0,0) and its permutations since any vector space is a subspace of itself.

4. Find the basis for the following subspaces of $\mathbb{R}^5$

(a) $\mathsf{W}_1 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 : a_1 - a_3 - a_4 = 0\}$

(b) $\mathsf{W}_2 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 : a_2 = a_3 = a_4, a_1 + a_5 = 0\}$

Solution:

(a) {(0,1,0,0,0),(1,0,1,0,0),(1,0,0,1,0),(0,0,0,0,1)}

(b) {(0,1,1,1,0),(-1,0,0,0,1)}

5. Do the polynomials $x^3 - 2x^2 + 1$, $4x^2 - x + 3$, $3x - 2$ generate $\mathsf{P}_3(\mathbb{R})$? where $\mathsf{P}_3(\mathbb{R})$ is the set of all polynomials having degree ≤ 3

Solution: None of the linear combinations of the given the polynomials produce $x \in \mathsf{P}_3(\mathbb{R})$ $\therefore$ they can't generate $\mathsf{P}_3(\mathbb{R})$

6. Prove that any rank 1 matrix has the form $\mathbf{A} = \mathbf{u}\mathbf{v}^T$ = column times row.

Solution: In rank 1 matrix, every row is a multiple of the first row, so the row space is one-dimensional. In fact, we can write as the whole matrix as *the product of a column vector* and *a row vector*.

Example: $\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$

7. Let V be a finite dimensional vector space and let $\mathcal{S}$ be a spanning subset of V. Prove that there exists a subset of $\mathcal{S}$ that is the basis for V.

Solution:

Here, we also provide a procedure to construct the basis $\mathcal{B}$ from $\mathcal{S}$. The procedure is as follows:

(a) Let $\mathcal{B} = \emptyset$ initially.

(b) If an element $\mathbf{u} \in \mathcal{S}$ is s.t $\{\mathbf{u}\} \cup \mathcal{B}$ is linearly independent, then add $\mathbf{u}$ into $\mathcal{B}$.

(c) Repeat (b) till linear independence of $\mathcal{B}$ fails.

Hence, by construction $\mathcal{B}$ is a linearly independent subset of $\mathcal{S}$. Now it suffices to show that $\mathcal{B}$ spans V, i.e., span$(\mathcal{B})$ = V. It is given that span$(\mathcal{S})$ = V.

Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ and $\mathcal{B} = \{\mathbf{v}_1, ..., \mathbf{v}_m\}$ $(m \leq n)$. This implies $\mathbf{v}_j = \sum\limits_{i=1}^{m} a_i^{(j)} \mathbf{v}_i$, $\forall j = m+1, ..., n$ and for some constants $a_1^{(j)}, ..., a_m^{(j)}$. By construction of $\mathcal{B}$, span$(\mathcal{B}) \subset$ span$(\mathcal{S})$. Now let $\mathbf{w} \in \text{span}(\mathcal{S}) \Rightarrow \mathbf{w} = \sum\limits_{k=1}^{n} b_k \mathbf{v}_k = \sum\limits_{k=1}^{m} b_k \mathbf{v}_k + \sum\limits_{k=m+1}^{n} b_k \Big(\sum\limits_{i=1}^{m} a_i^{(k)} \mathbf{v}_i \Big) \Rightarrow \mathbf{w} \in$ span$(\mathcal{B})$. Thus, span$(\mathcal{B})$ = span$(\mathcal{S})$ = V. Hence, proved.

8. Prove that number of basis vectors of a vector space is unique.

Solution: Let $\mathbf{v}_1, ..., \mathbf{v}_m$ and $\mathbf{w}_1, ..., \mathbf{w}_n$ are both bases for the same vector space. Suppose there are more $\mathbf{w}_j$'s than $\mathbf{v}_i$'s $(n > m)$. Since $\mathbf{v}_i$'s form a basis, they must span the space. Every $\mathbf{w}_j$ can be written as a combination of the $\mathbf{v}_i$'s: If $\mathbf{w}_j = a_{1j}\mathbf{v}_1 + ... + a_{mj}\mathbf{v}_m$.

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & ... & \mathbf{w}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & ... & \mathbf{v}_m \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} = \mathbf{VA}.$$

We know the shape of $\mathbf{A}$ is m by n. since $n > m$, There is a nonzero solution to $\mathbf{Ax} = \mathbf{0}$. Then $\mathbf{VAx} = \mathbf{0} \Rightarrow \mathbf{Wx} = \mathbf{0}$. A combination of the $\mathbf{w}_j$'s gives zero! The $\mathbf{w}_j$'s could not be a basis, which is a contradiction. Similarly, $(m > n)$ also will fail. Hence, number of basis vectors of a vector space is unique $(m = n)$.

9. Let $\mathbf{A}$ be an $m \times n$ matrix. Prove that the sum of dimensions of the column space and null space of $\mathbf{A}$ equals n.

Solution:

Note that $\mathbf{A}$ transforms vectors from $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$. Null space of $\mathbf{A} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{Ax} = \mathbf{0}_m\}$ ($\mathbf{0}_m$ is an $m \times 1$ all zero vector). Clearly, null space of $\mathbf{A} \subset \mathbb{R}^n$. Also, it is also a subspace of $\mathbb{R}^n$. Let $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ $(k \leq n)$ be the basis for null space of $\mathbf{A}$. Then, it can be extended to the basis of $\mathbb{R}^n$ as $\{\mathbf{v}_1, ..., \mathbf{v}_k, \mathbf{v}_{k+1}, ..., \mathbf{v}_n\}$. If we prove that

$\{\mathbf{Av}_{k+1}, ..., \mathbf{Av}_n\}$, which are all $m \times 1$ vectors, is the basis for column space of $\mathbf{A}$, then the final result follows as,

$$\dim(\text{column space of } \mathbf{A}) + \dim(\text{null space of } \mathbf{A}) = n - k + k = n.$$

To prove: $\mathcal{S} = \{\mathbf{Av}_{k+1}, ..., \mathbf{Av}_n\}$ *is the basis of the column space of* $\mathbf{A}$.

(i) **Linear Independence:**
Let $\mathcal{S}$ be linearly dependent. Then, there exists constants $a_1, ..., a_{n-k}$ such that $\sum_{i=1}^{n-k} a_i \mathbf{Av}_{k+i} = \mathbf{0}_m$ with not all a_i's equal to zero. Now, $\mathbf{A}\Big(\sum_{i=1}^{n-k} a_i \mathbf{v}_{k+i}\Big) = \mathbf{0}_m \Rightarrow \sum_{i=1}^{n-k} a_i \mathbf{v}_{k+i} \in$ null space of $\mathbf{A}$. Then, $\exists b_1, ..., b_k$ s.t, $\sum_{i=1}^{n-k} a_i \mathbf{v}_{k+i} = \sum_{j=1}^{k} b_j \mathbf{v}_j$. This contradicts the fact that $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ are linearly independent. Thus, the assumption that $\mathcal{S}$ is linearly dependent is incorrect $\Rightarrow \mathcal{S}$ is linearly independent.

(b) **$\mathcal{S}$ spans column space of "A":**
Let $\mathbf{x} \in$ column space of $\mathbf{A}$. Then, $\exists \mathbf{y} \in \mathbb{R}^n$ s.t. $\mathbf{Ay} = \mathbf{x}$. But $\mathbf{y} = \sum_{i=1}^{n} c_i \mathbf{v}_i$, for some constants c_i's (basis expansion in $\mathbb{R}^n$). So, $\mathbf{x} = \mathbf{Ay} = \mathbf{A}\Big(\sum_{i=1}^{n} c_i \mathbf{v}_i\Big) = \mathbf{A}\Big(\sum_{i=1}^{k} c_i \mathbf{v}_i\Big) + \mathbf{A}\Big(\sum_{i=k+1}^{n} c_i \mathbf{v}_i\Big) = \mathbf{0}_m + \mathbf{A}\Big(\sum_{i=k+1}^{n} c_i \mathbf{v}_i\Big) = \mathbf{A}\Big(\sum_{i=k+1}^{n} c_i \mathbf{v}_i\Big) \Rightarrow \mathbf{x} \in \text{span}(\mathcal{S}) \Rightarrow$ column space of $\mathbf{A} \subset \text{span}(\mathcal{S})$. But $\text{span}(\mathcal{S}) \subset$ column space of $\mathbf{A}$. So $\mathcal{S}$ spans the column space of $\mathbf{A}$.

From above discussions, we proved the desired result.

10. Let $\mathbf{A}$ be an $n \times n$ invertible conjugate symmetric matrix, i.e., $\mathbf{A}^H = (\mathbf{A}^*)^T = \mathbf{A}$ ($*$ - denotes conjugation) and $\mathbf{x}$ be an $n \times 1$ vector. The following procedure guides you to find the inverse of $\mathbf{A} + \mathbf{xx}^H$.

    (a) For an arbitrary $\mathbf{y}$, consider the equation,

    $$(\mathbf{A} + \mathbf{xx}^H)\mathbf{z} = \mathbf{y}. \tag{1}$$

    Now finding inverse of $\mathbf{A} + \mathbf{xx}^H$ is equivalent to finding a $\mathbf{B}$ such that $\mathbf{z} = \mathbf{By}$. Pre-multiply both sides of equation (1) by $\mathbf{A}^{-1}$ and obtain

    $$\mathbf{z} = \mathbf{A}^{-1}\mathbf{y} - \mathbf{A}^{-1}\mathbf{xx}^H\mathbf{z}. \tag{2}$$

    (b) Pre-multiply both sides of equation (2) by $\mathbf{x}^H$ and then solve for $\mathbf{x}^H\mathbf{z}$ in terms of $\mathbf{x}, \mathbf{A}, \mathbf{y}$.

    (c) Substitute into equation (1) and manipulate to bring into the desired form $\mathbf{z} = \mathbf{By}$. Observe what $\mathbf{B}$ is.

Solution: Consider equation (2). Now, on pre-multiply both sides of this equation by$\mathbf{x}^H$, we get, $\mathbf{x}^H\mathbf{z} = \mathbf{x}^H\mathbf{A}^{-1}\mathbf{y} - \mathbf{x}^H\mathbf{A}^{-1}\mathbf{x}\mathbf{x}^H\mathbf{z} \Rightarrow \mathbf{x}^H\mathbf{z} = \mathbf{x}^H\mathbf{A}^{-1}\mathbf{y}/(1+\mathbf{x}^H\mathbf{A}^{-1}\mathbf{x})$. Let $\tilde{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{x}$. Substitute this in equation (1). We get,

$$\begin{aligned}
&\mathbf{A}\mathbf{z} + \mathbf{x}\mathbf{x}^H\mathbf{z} = \mathbf{y} \\
\Rightarrow &\mathbf{A}\mathbf{z} + \mathbf{x}\tilde{\mathbf{x}}^H\mathbf{y}/(1+\mathbf{x}^H\tilde{\mathbf{x}}) = \mathbf{y} \\
\Rightarrow &\mathbf{A}\mathbf{z} = \left(\mathbf{I} - \frac{\mathbf{x}\tilde{\mathbf{x}}^H}{1+\mathbf{x}^H\tilde{\mathbf{x}}}\right)\mathbf{y} \\
\Rightarrow &\mathbf{z} = \left(\mathbf{A}^{-1} - \frac{\tilde{\mathbf{x}}\tilde{\mathbf{x}}^H}{1+\mathbf{x}^H\tilde{\mathbf{x}}}\right)\mathbf{y}.
\end{aligned}$$

Thus, inverse of $\mathbf{A} + \mathbf{x}\mathbf{x}^H$ is $\mathbf{A}^{-1} - \frac{\tilde{\mathbf{x}}\tilde{\mathbf{x}}^H}{1+\mathbf{x}^H\tilde{\mathbf{x}}}$, with $\tilde{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{x}$.