EE5120 Linear Algebra: Tutorial 1, July-Dec 2017-18

1. Solve the following sets of linear equations using Gaussian elimination

   (a)

$$\begin{aligned} 2x_1 - 2x_2 - 3x_3 &= -2 \\ 3x_1 - 3x_2 - 2x_3 + 5x_4 &= 7 \\ x_1 - x_2 - 2x_3 - x_4 &= -3 \end{aligned}$$

   (b)

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 5 \\ x_1 + 4x_2 - 3x_3 - 3x_4 &= 6 \\ 2x_1 + 3x_2 - x_3 + 4x_4 &= 8 \end{aligned}$$

2. State True or False for each of the following with proper justification:
   (a) An elementary matrix is always a square matrix.
   (b) The $n \times n$ identity matrix is an elementary matrix.
   (c) Product of two elementary matrices (each of appropriate dimensions) is an elementary matrix.
   (d) Sum of two elementary matrices of same dimension is also an elementary matrix.
   (e) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then A can be obtained by performing an elementary row operation on B.
   (f) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then B can also be obtained by performing an elementary column operation on A.

3. What three elimination matrices E_{21}, E_{31}, E_{32} put A into upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1}, E_{31}^{-1} and E_{21}^{-1} to factor A into LU where $L = E_{32}^{-1}E_{31}^{-1}E_{21}^{-1}$. Find L and U:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

4. Find L and U for the nonsymmetric matrix

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}$$

   Find the four conditions on a, b, c, d, r, s, t to get $A = LU$ with four pivots.

5. (a) The equation of the line through the following pair of points $(3, -2, 4)^T$ and $(-5, 7, 1)^T$ in $\mathbb{R}^3$ is

$$(x, y, z)^T = (3, -2, 4)^T + t(........, ..., ...)^T$$

   where $t \in \mathbb{R}$.

(b) The equation of the plane through the following set of points $(2, -5, -1)^T$, $(0, 4, 6)^T$ and $(-3, 7, 1)^T$ in $\mathbb{R}^3$ is

$$(x, y, z)^T = (2, -5, -1)^T + s(........, ..., ...)^T + t(........, ..., ...)^T$$

where $s, t \in \mathbb{R}$.

6. If (a, b) is a multiple of (c, d) with $abcd \neq 0$, show that (a, c) is a multiple of (b, d). Thus show that if a matrix has dependent rows, then it has dependent columns.

7. Working with inverse matrices:

(a) Suppose that $A \in \mathbb{R}^{n \times n}$ is a square invertible matrix and $u, v \in \mathbb{R}^n$ are column vectors. Prove that

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u}$$

provided $1 + v^T A^{-1} u \neq 0$. This formula goes by the name of Sherman-Morrison.

(b) The practical use of this formula? Let's say that you have been solving matrix equations of the form $Ax = b$ and the system description (embedded in A) changed slightly from A to $A' = A + uv^T$. This formula allows you to use your previous method (e.g. the LU factors of A) in solving for a new $A'x' = b$. Show that a new LU decomposition of A' is unnecessary.