PROBLEM SET-6-SOLUTIONS

EE6340

① preprocessing the output :-

ⓐ Since, $\tilde{y} = g(y)$, we can say $X \to Y \to \tilde{Y}$ forms a markov chain.

By data processing Inequality,

$$I(X;Y) \geq I(X;\tilde{Y})$$

Let $\tilde{p}(x)$ be the distribution that maximizes $I(X;\tilde{Y})$

$$C = \max_{p(x)} I(X;Y) \geq I(X;Y)_{p(x)=\tilde{p}(x)} \geq I(X;\tilde{Y})_{p(x)=\tilde{p}(x)}$$

$$= \max_{p(x)} I(X;\tilde{Y})$$

$$= \underline{\tilde{C}}.$$

Therefore, the statistician is proved to be wrong and preprocessing the output doesnot increase capacity.

ⓑ we have equality In the above derivation only if $X \to \tilde{Y} \to Y$ also form a markov chain.

(2) Additive Noise Channel :-

$$y = x + Z$$

$x \in \{0, 1\}$, $Z \in \{0, a\}$

Based on the value of 'a', output can be '4' or '3' values.

ie, for $a \neq \pm 1$,

In this case 'y' can take $\{0, 1, a, 1+a\}$ which are distinct.

knowing 'y' there is no uncertainity in X and hence $H(X/Y) = 0$.

$\therefore$ $\max I(X;Y) = \max H(X) = 1$ on $P(x) = 1/2$.

$a = +1$

In this case Y can take '3' values $\{0, 1, 2\}$.

This is similar to erasure channel and capacity of this channel is $1/2$ Bit/Transmission.

$a = -1$

y can take $\{0, 1, -1\}$, and channel is once again erasure.

$$C = 1/2 \text{ Bit/Transmission.}$$

(3) (a) $$Y = X + Z \pmod{11}$$

$$Z = \begin{cases} 1 \\ 2 \\ 3 \end{cases} \text{ with probability } 1/3 \text{ each.}$$

$$H(Y/X) = H(Z/X) = H(Z) = \log 3.$$

$H(Z/X) = H(Z)$ since 'Z' is independent of distribution of 'X'.

$$C = \max_{P(x)} I(X;Y)$$

$$= \max_{P(x)} H(Y) - H(Y/X)$$

$$= \max_{P(x)} H(Y) - \log 3.$$

$H(Y)$ is maximum, when 'Y' is uniformly distributed, which occurs when 'X' is uniform (By symmetry)

$$\therefore \quad C = \log 11 - \log 3$$

$$= \log(11/3) \text{ Bits / Transmission.}$$

(b) This capacity is acheived by uniform distribution on the Input.

④ Capacity of the product channel (two channels at a time), is the problem to find $I(x_1, x_2; y_1, y_2)$ and its maximizing distribution $p(x_1, x_2)$.

From, the joint distribution

$$p(x_1, x_2, y_1, y_2) = p(x_1, x_2)\, p(y_1/x_1)\, p(y_2/x_2)$$

we can say,

$$Y_1 \rightarrow X_1 \rightarrow X_2 \rightarrow Y_2 \text{ form a markov chain.}$$

$$\therefore \quad I(x_1, x_2; y_1, y_2) = H(Y_1, Y_2) - H(Y_1; Y_2/x_1, x_2)$$

$$= H(Y_1, Y_2) - H(Y_1/x_1, x_2) - H(Y_2/x_1, x_2) \quad [\text{markov}]$$

$$= H(Y_1, Y_2) - H(Y_1/x_1) - H(Y_2/x_2) \quad [\text{from markov}]$$

$$\leq H(Y_1) + H(Y_2) - H(Y_1/x_1) - H(Y_2/x_2) \quad [H(Y_1, Y_2) \leq H(Y_1) + H(Y_2)]$$

$$= I(X_1; Y_1) + I(X_2; Y_2)$$

$$C = \max_{p(x_1, x_2)} I(x_1 x_2 ; y_1 y_2)$$

$$\leq \max_{p(x_1, x_2)} I(x_1 ; y_1) + \max_{p(x_1, x_2)} I(x_2 ; y_2)$$

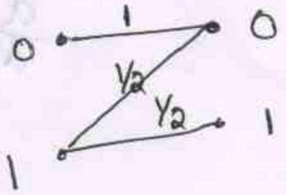
$$= \max_{p(x_1)} I(x_1 ; y_1) + \max_{p(x_2)} I(x_2 ; y_2)$$

$$= C_1 + C_2$$

with equality if and only if $p(x_1, x_2) = p^*(x_1) \cdot p^*(x_2)$

and $p^*(x_1)$, $p^*(x_2)$ are the probabilities that maximize

C_1 and C_2.



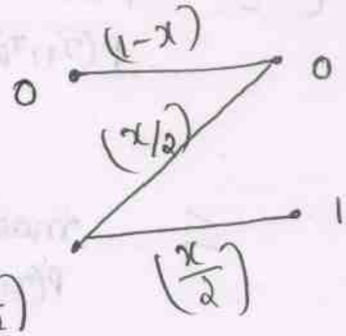
(5) Z-channel :- Let $Pr[X = 1] = x$;

$$H(Y/X) = Pr(x=0)\, H(Y/x=0) + Pr(x=1)\, H(Y/x=1)$$

$$= (1-x) \cdot 0 + x \cdot 1$$

$$= x$$

$$H(Y) = H\left(Pr(Y=1)\right)$$



$$= H\left(x/2\right)$$

$$= -\frac{x}{2}\log\left(\frac{x}{2}\right) - \left(1-\frac{x}{2}\right)\log\left(1-\frac{x}{2}\right)$$

$$\therefore \quad I(X;Y) = H(Y) - H(Y/X)$$

$$= H\left(x/2\right) - x$$

To find the maximum $I(X;Y)$ differentiate w.r.t. x.

Final equation would be,

$$\frac{d}{dx} I(X;Y) = \frac{1}{2}\log_2 \frac{1-\frac{x}{2}}{\frac{x}{2}} - 1 = 0$$

~~For~~ $x = 2/5$ ~~[at $x = 0$ or 1, $I(X;Y) = 0$], so, $x = 2/5$ is chosen~~

$\therefore$ The capacity of the Z-channel is bits is

$$C = H\left(1/5\right) - 2/5 = 0.322 \text{ Bits/Transmission}$$