

EE6340: Information Theory

Problem Set 8

1. *Differential entropy.* Evaluate the differential entropy $h(X) = -\int f \ln f$ for the following:
 - (a) The exponential density, $f(x) = \lambda e^{-\lambda x}, x \geq 0$.
 - (b) The Laplace density, $f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}$.
 - (c) The sum of X_1 and X_2 , where X_1 and X_2 are independent normal random variables with means μ_i and variances $\sigma_i^2, i = 1, 2$.

2. *Concavity of determinants.* Let K_1 and K_2 be two symmetric nonnegative definite $n \times n$ matrices. Prove the results of Ky Fan:

$$|\lambda K_1 + \bar{\lambda} K_2| \geq |K_1|^\lambda |K_2|^{\bar{\lambda}}, \text{ for } 0 \leq \lambda \leq 1, \bar{\lambda} = 1 - \lambda, \quad (1)$$

where $|K|$ denotes the determinant of K

Hint: Let $Z = X_\theta$, where $X_1 \sim \mathcal{N}(0, K_1), X_2 \sim \mathcal{N}(0, K_2)$ and $\theta = \text{Bernoulli}(\lambda)$. Then use $H(Z|\theta) \leq H(Z)$.

3. *Mutual information for correlated normals.* Find the mutual information $I(X; Y)$, where

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}\right) \quad (2)$$

Evaluate $I(X; Y)$ for $\rho = 1, \rho = 0$, and $\rho = -1$, and comment.

4. *Uniformly distributed noise.* Let the input random variable X to a channel be uniformly distributed over the interval $-1/2 \leq x \leq +1/2$. Let the output of the channel be $Y = X + Z$, where the noise random variable is uniformly distributed over the interval $-a/2 \leq x \leq +a/2$, and independent of the input.
 - (a) Find $I(X; Y)$ as a function of a .
 - (b) For $a = 1$ find the capacity of the channel when the input X is peak-limited; that is, the range of X is limited to $-1/2 \leq x \leq +1/2$. What probability distribution on X maximizes the mutual information $I(X; Y)$?
5. *The two-look Gaussian channel.*



Consider the ordinary Shannon Gaussian channel with two correlated looks at X , i.e., $Y = (Y_1, Y_2)$, where

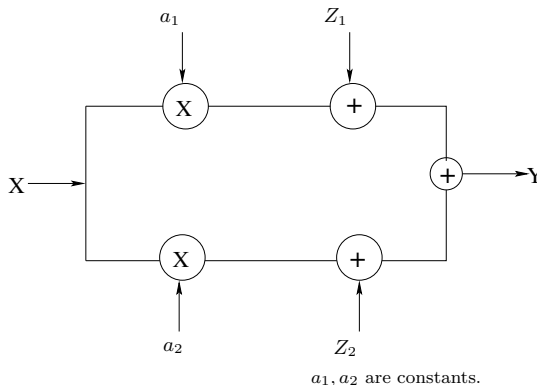
$$\begin{aligned} Y_1 &= X + Z_1 \\ Y_2 &= X + Z_2 \end{aligned}$$

with a power constraint P on X , and $(Z_1, Z_2) \sim \mathcal{N}_2(0, K)$, where

$$K = \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix} \quad (3)$$

Find the capacity C for (a) $\rho = 1$, (b) $\rho = 0$, (c) $\rho = -1$.

6. *Multipath Gaussian channel.* Consider a Gaussian noise channel of power constraint P where the signal takes two different paths and the received noisy signals are added at the antenna.

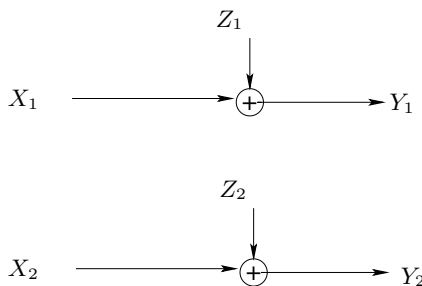


- (a) Find the capacity if Z_1 and Z_2 are jointly Gaussian with covariance matrix

$$K_Z = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix} \quad (4)$$

- (b) What is the capacity for $\rho = 0, \rho = -1, \rho = 1$?

7. *Parallel Gaussian channels.* Consider the following parallel Gaussian channel



where $Z_1 \sim \mathcal{N}(0, N_1)$ and $Z_2 \sim \mathcal{N}(0, N_2)$ are independent Gaussian random variables and $Y_i = X_i + Z_i, i = 1, 2$. We have a total cost constraint $\beta_1 P_1 + \beta_2 P_2 \leq \beta$, where P_i is the power allocated to the i 'th channel and β_i is the cost per unit power in that channel. We fix β_1 and β_2 and wish to allocate power $P_1 \geq 0, P_2 \geq 0$ to the two channels subject to the cost constraint to maximize the capacity.

- (a) For what value of β does the channel stop acting like a single channel and start acting like a pair of channels?
 (b) Evaluate the capacity and find P_1, P_2 that achieve capacity for $\beta_1 = 1, \beta_2 = 2, N_1 = 3, N_2 = 2$ and $\beta = 10$.