Efficient Quantum Algorithm for Computing the Circumference of Infrastructures

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Outline

Infrastructures

Quantum Information

Quantum Algorithm
- Preprocessing
- Quantum part of the algorithm
- Classical part of the algorithm

Infrastructures

An infrastructure of circumference R is a pair (X, d) where

- $X = \{x_0, x_1, \ldots, x_{m-1}\}$
- $d : X \hookrightarrow \mathbb{R}$
  $d(x_0) = 0 < d(x_1) < \cdots < d(x_{m-1}) < R$

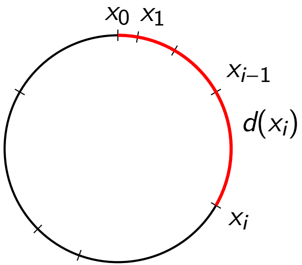


Figure: Visualizing an infrastructure

Functions on Infrastructures

Baby-step : $\mathrm{bs} : X \to X$

$$\mathrm{bs}(x_i) = \begin{cases} x_{i+1} & 0 \leq i < m-1 \\ x_0 & i = m-1 \end{cases} \quad (1)$$

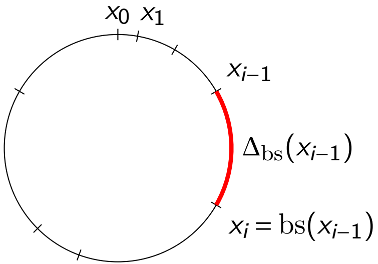


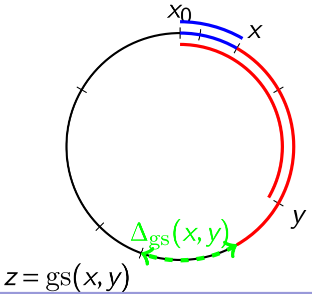
Figure: Baby-step

Functions on Infrastructures

Giant-step : $\mathrm{gs}: X \times X \to X$

Operational interpretation of $\mathrm{gs}(x, y)$

- From x move a distance of $d(y)$ along the circle.
- Find the element $z \in X$ that is immediately "after" $d(x) + d(y)$.
- $\Delta_{\mathrm{gs}}(x, y) = d(z) - d(x) - d(y) \bmod R$.



Computational Problems

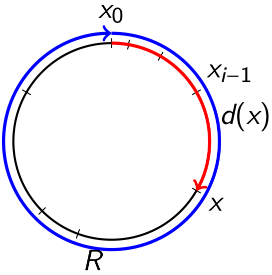


Figure: Computational problems of an infrastructure

- Compute an m-bit approximation of the circumference R.
- Given an element x, compute an m-bit approximation of $d(x)$.

Example

A cyclic group $G = \langle g \rangle = \{1, g, g^2, \ldots, g^{m-1}\}$. Distance function

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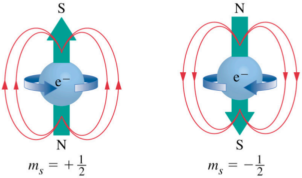
Discrete logarithm problem: Given g^i, compute i.

Qubits i.e. Quantum Bits

Qubits are 2-state quantum systems



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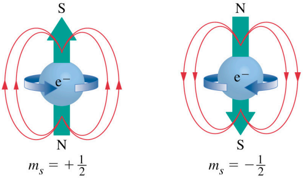

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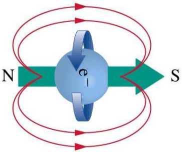

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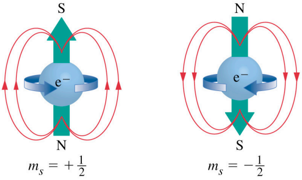

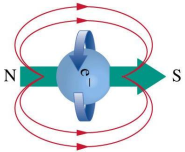

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The state of n qubits is a unit vector in $\mathbb{C}^{2^n} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n}$.

$$\left|\psi\right\rangle = \sum_{x_i \in \mathbb{F}_2} \alpha_{x_1,\ldots,x_n} |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle; \qquad \sum_{x_i \in \mathbb{F}_2} |\alpha_{x_1,\ldots,x_n}|^2 = 1.$$

Qubits—Measurement

In general observing (measuring) a quantum state changes the state.

$$\alpha|0\rangle + \beta|1\rangle \overset{\text{Observe}}{\longmapsto} \begin{cases} |0\rangle & Pr(|0\rangle) = |\alpha|^2 \\ |1\rangle & Pr(|1\rangle) = |\beta|^2 \end{cases}$$

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More generally if we observe some qubits of a state, it "collapses" the state.

$$\sum_i \sum_j a_{i,j} |i\rangle_A |j\rangle_B \rightarrow \sum_i a_i |i\rangle |j_B\rangle_B$$

No cloning and entanglement

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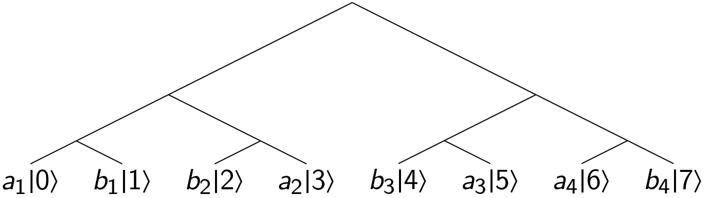
Entanglement: Consider the state

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \neq (a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle)$$

Entangled states exhibit non-local correlations.

Quantum Parallelism and Interference

A quantum computer takes many computational paths simultaneously.



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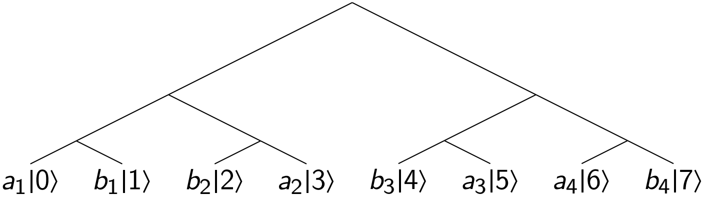


Quantum computer interferes different computational paths so that only the desirable final states survive with high probabilities.

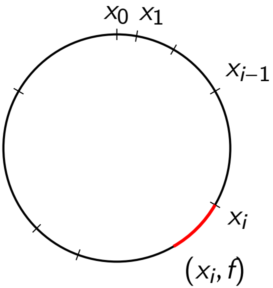
The Road Ahead—Some Obstacles and the Strategy

Some obstacles

- We must be able to compute efficiently within the infrastructure, i.e., compute $\mathrm{bs}(x)$, $\mathrm{gs}(x,y)$, $\Delta_{\mathrm{bs}}(x)$ and $\Delta_{\mathrm{gs}}(x,y)$ (without the knowledge of R).
- The distances could be transcendental, but we assume only finite precision arithmetic.

Imposing a group structure

In order to be able to compute efficiently within an infrastructure, we embed into a circle group.



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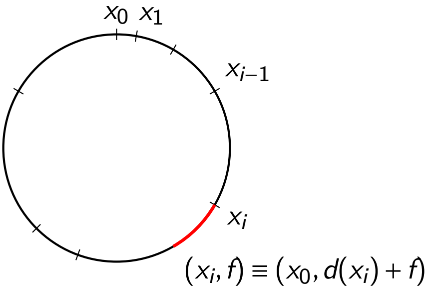

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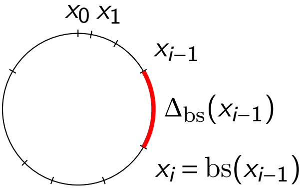
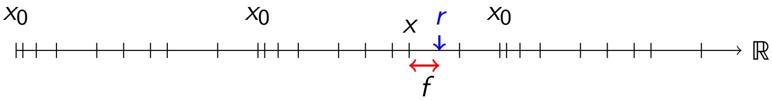
- Discretize the approximate version of h still preserving "approximate periodicity".
- Use quantum Fourier transform to estimate the period R by finding an integer close to R.
- Refine the approximation to the desired degree of precision using classical post processing.

Setting Up a Periodic Function, $h : \mathbb{R} \to \mathbb{R}/R\mathbb{Z}$



Unwrap the circle and place it on $0 \cup \mathbb{R}^+$ as shown below.



$h(r) = (x, f)$, where x is the nearest element of X to the left of r and $f = r - d(x) \bmod R$.

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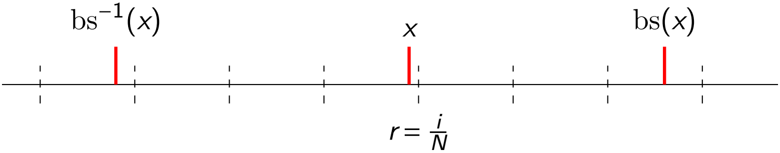

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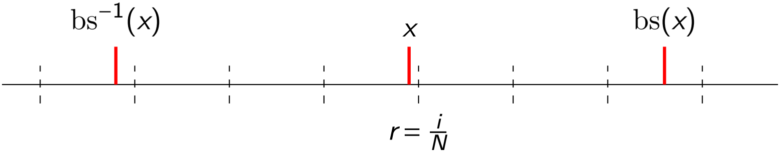

- Shift the evaluation points so that they are not too close to the elements of X.

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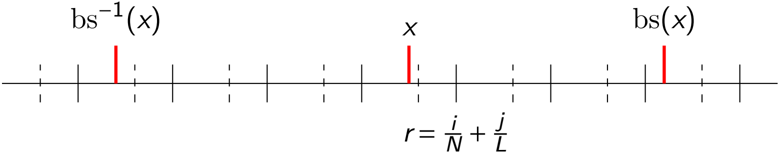

Setting a Pseudo-Periodic State

We choose a sufficiently large integer $q \geq 2R^2N^2$.

— Prepare two quantum registers in the following state

$$|\psi\rangle \mapsto \sum_{j=0}^{q-1} |j\rangle_1 |0\rangle_2 \quad (2)$$

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- Measure the second register.

$$|\psi\rangle \mapsto \sum_{j\in\mathscr{J}} |o + \lfloor jRN \rceil\rangle_1 |h_N(o))\rangle_2, \quad \mathscr{J} \subseteq \{0, 1, \ldots, \lfloor q/NR \rfloor - 1\} \tag{4}$$

We call these states pseudo-periodic states.

Quantum Fourier Transform

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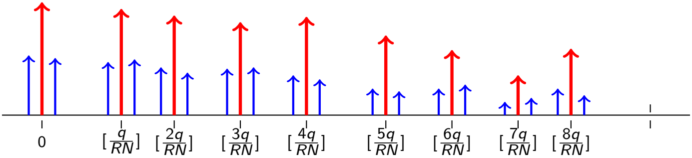
$$|k\rangle \overset{QFT}{\mapsto} \frac{1}{\sqrt{q}} \sum_{j=0}^{q-1} e^{i2\pi kj/q} |j\rangle \quad (5)$$

$$|\psi\rangle \overset{QFT}{\mapsto} \frac{1}{\sqrt{q|\mathscr{J}|}} \sum_{\ell=0}^{q-1} \sum_{j\in\mathscr{J}} e^{i2\pi\ell\lfloor jRN\rceil/q} |\ell\rangle \quad (6)$$

Extracting Periodicity via Quantum Fourier Transform

A q-point QFT of the first register gives the following state[1].



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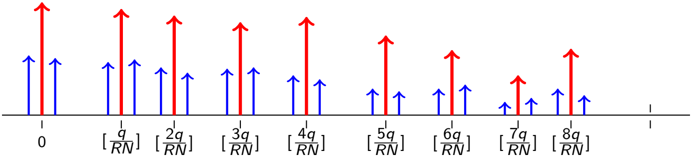

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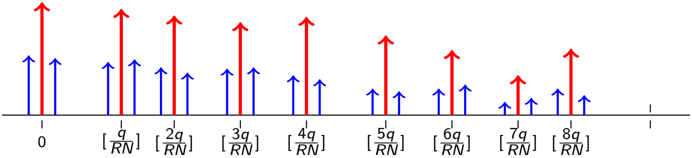

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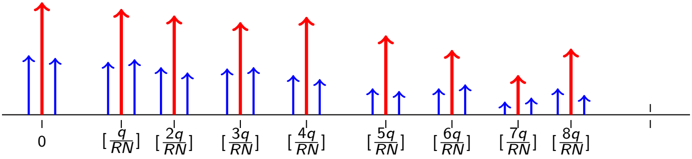

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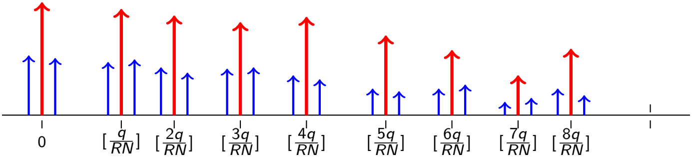

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- In order to extract the period we need to measure a term $\ell = [mq/NR]$.
- Such a state occurs with high probability provided the state was "sufficiently" periodic and q is large.
- Two such measurements can be used to approximate an integer that is close to RN using continued fractions.

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Classical Postprocessing

— Observe that $\ell = [\frac{mq}{NR}]$ that there are two unknowns, m and R. So one measurement is not sufficient.

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- Compute $h(\overline{NR}/N)$ and if this is in the neighbourhood of x_0, then $\overline{NR}/N$ is within a unit of R or its multiples.
- Successively improve the estimate of R using a binary search procedure with the baby-step.

Improvements

Proposed algorithm

- Generalizes Hallgren's algorithm for number fields, and can be used to solve Pell's equation $x^2 - dy^2 = 1$.
- Uses a tighter analysis and presents a technical result of larger applicability.
- Has lower complexity, polynomial speedup over Hallgren's algorithm for Pell's equation.
- Success probability $= \Omega(1)$, in contrast Hallgren's algorithm which can only guarantee a success probability $\Omega(1/\log N^4 R^4)$.

Summary

- Polynomial time quantum algorithms for infrastructures. PS, Wocjan arXiv:1106.5347
  - Computing the circumference of the infrastructures.
  - Computing the generalized discrete logarithms.
- When specialized to cases such as the quadratic number fields, the proposed algorithms have
  - Lower complexity.
  - Higher success probability.

Questions?

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Thank you!