Majority logic decoding

Note Title

Ex: $(7,3,4)$ code $\quad x^7+1=(x+1)(x^3+x+1)(x^3+x^2+1)$

$$g(x)=(1+x)(1+x+x^3)$$
$$=1+x+x^3+x+x^2+x^4$$
$$=1+x^2+x^3+x^4$$

$$G=\begin{bmatrix}1&0&1&1&1&0&0\\0&1&0&1&1&1&0\\0&0&1&0&1&1&1\end{bmatrix}\qquad H=\begin{bmatrix}1&1&0&1&0&0&0\\0&1&1&0&1&0&0\\0&0&1&1&0&1&0\\0&0&0&1&1&0&1\end{bmatrix}$$

1	1	0	1	0	0	0	✓	$c_0+c_1+c_3=0$
1	0	1	1	1	0	0		
1	1	1	0	0	1	0		
1	1	0	0	1	0	1		
1	0	0	0	1	1	0	✓	$c_0+c_4+c_5=0$
1	1	1	1	1	1	1		
1	0	1	0	0	0	1	✓	$c_0+c_2+c_6=0$
1	0	0	1	0	1	1		

$$\underline{r}=[r_0\ r_1\ r_2\ r_3\ r_4\ r_5\ r_6]$$

$$\hat{c}_0^{(1)}=r_1+r_3,\quad \hat{c}_0^{(2)}=r_4+r_5,\quad \hat{c}_0^{(3)}=r_2+r_6$$

$$\hat{c}_0=\mathrm{maj}\{\hat{c}_0^{(1)},\hat{c}_0^{(2)},\hat{c}_0^{(3)}\}$$

$$\underline{e} = [e_0 \; e_1 \; e_2 \; e_3 \; e_4 \; e_5 \; e_6]$$

$$\underline{r} = \underline{c} + \underline{e}$$

$$s_1 = r_0 + r_1 + r_3, \quad s_2 = r_0 + r_4 + r_5$$

$$s_1 = e_0 + e_1 + e_3, \quad s_2 = e_0 + e_4 + e_5$$

$$\hat{e}_0 = \text{maj}\{s_1, s_2\}, \text{ tie} \Rightarrow \hat{e}_0 = 0$$

Result: J orthogonal parity checks (even or odd)

$$\Downarrow$$

$\lfloor \frac{J}{2} \rfloor$ errors are correctable

(7,4,3) Hamming code

$$g(x) = 1 + x + x^3$$

Parity-checks orthogonal on $\{0,2\}$.

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$c_0 + c_2 + c_3 + c_4 = 0$

$$\begin{matrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{matrix}$$

$c_0 + c_2 + c_1 + c_5 = 0$

$\{0,1\}$

$c_0 + c_1 + c_4 + c_6 = 0$

$$S_1 = r_0 + r_2 + r_3 + r_4 = e_0 + e_2 + e_3 + e_4$$

$$S_2 = r_0 + r_1 + r_2 + r_5 = e_0 + e_1 + e_2 + e_5$$

$$S_3 = r_0 + r_1 + r_4 + r_6 = e_0 + e_1 + e_4 + e_6$$

$\hat{e}_0 + \hat{e}_2 = \text{maj}\{S_1, S_2\}$ (tie $\Rightarrow 0$)

$\hat{e}_0 + \hat{e}_1 = \text{maj}\{S_2, S_3\}$ "

$\hat{e}_0 = \text{maj}\{\hat{e}_0 + \hat{e}_2, \hat{e}_0 + \hat{e}_1\}$ "

Main result: $RM(r, m)$:

$(r+1)$-step majority logic decoding can correct $\left\lfloor \frac{1}{2}(2^{m-r} - 1) \right\rfloor$ errors.

field size

$\{0,1\}^m$ as a finite geometry: $EG(m, 2)$

Euclidean geometry dim

Vars: $v_1, v_2, \dots, v_m$ $\{0,1\}^m$: set of points

$(m-1)$-flat : subset of points that satisfies

or

hyperplane $\displaystyle\sum_{i=1}^{m} a_i v_i + b = 0$

$a_i \in \{0,1\}$ not all zero

$b \in \{0,1\}$

$(m-r)$-flat in $EG(m,2)$:

Set of points satisfying r equations (linear)

$$\sum_{j=1}^{m} a_{ij} v_j + b_i = 0$$

$$i = 1, 2, \ldots, r$$

Ex: $EG(3,2)$: $\{0,1\}^3$, vars: v_1, v_2, v_3

2-flat: $v_1 + v_2 + 1 = 0$ $\{010, 011, 100, 101\}$

1-flat: $v_1 + v_2 + 1 = 0$, $v_1 = 0$ $\{010, 011\}$