Lecture 6

Note Title 8/6/2008

Modulation and Optimum Receiver Principles

$$Y(t) = X(t) + N(t) \qquad \text{"waveform chanel"}$$

$$\Big\Downarrow \text{No loss}$$

$$\underline{Y} = \underline{X} + \underline{N} \qquad \text{"Vector channel"}$$

$$\underline{b} = [b_1 \ldots b_N] \longrightarrow \boxed{T_x} \longrightarrow X(t) = x_i(t) \quad \text{w.p. } 1/2^N$$

$$1 \leq i \leq 2^N$$

$$\text{supp}\{x_i(t)\} \subseteq [0, T] \qquad \text{Bit rate} = \frac{N}{T}$$

"Gram-Schmidt" on $\{x_i(t)\}$ to get M orthonormal basis $\{\phi_1(t), \ldots, \phi_M(t)\}$

$$\langle \phi_i(t), \phi_j(t) \rangle = \delta_{ij}$$

Gram-Schmidt:

$$\phi_1(t) = \frac{x_1(t)}{\|x_1(t)\|_2}$$

$$\tilde{\phi}_2(t) = x_2(t) - \langle x_2(t), \phi_1(t) \rangle \phi_1(t)$$

$$\phi_2(t) = \frac{\tilde{\phi}_2(t)}{\|\tilde{\phi}_2(t)\|_2}$$

$$\tilde{\phi}_3(t) = x_3(t) - \langle x_3(t), \phi_1(t) \rangle \phi_1(t) - \langle x_3(t), \phi_2(t) \rangle \phi_2(t)$$

$$\phi_3(t) = \frac{\tilde{\phi}_3(t)}{\|\tilde{\phi}_3(t)\|_2}$$

& so on

Now

$$x_i(t) = \langle x_i(t), \phi_1(t) \rangle \phi_1(t) + \langle x_i(t), \phi_2(t) \rangle \phi_2(t) + \cdots + \langle x_i(t), \phi_M(t) \rangle \phi_M(t)$$

$$x_{ij} = \langle x_i(t), \phi_j(t) \rangle \qquad 1 \le i \le 2^N, \quad 1 \le j \le M$$

$$x_i(t) = \sum_{j=1}^{M} x_{ij} \phi_j(t), \quad 1 \le i \le 2^N$$

Signal $x_i(t) \longleftrightarrow \underline{x}_i = [x_{i1} \; x_{i2} \; \cdots \; x_{iM}]^T \in \mathbb{C}^M$

$X(t) \longrightarrow \underline{X}$ $\quad Pr(\underline{X} = \underline{x}_i) = \frac{1}{2^N}, \quad 1 \le i \le 2^N$

CT RP $\qquad$ Random vector

$$\underline{X} = [X_1 \ X_2 \cdots X_m]$$

X_i : random variable

$$X(t) = \sum_{j=1}^{m} X_j \phi_j(t)$$

Noise Signal: $N(t)$: white Gaussian RP

$$S_N(f) = \frac{N_0}{2}$$

Sample function

$$n(t) = \overbrace{\sum_{j=1}^{M} \langle n(t), \phi_j(t) \rangle \phi_j(t)}^{n_1(t)} + n_2(t)$$

$$N(t) \longrightarrow \boxed{\int (\cdot)\, \phi_i(t)} \longrightarrow N_j \sim \text{Normal } N\left(0, \frac{N_0}{2}\right)$$

$\{n_1(t)\}$: Random process

$$N_1(t) = \sum_{j=1}^{M} N_j \, \phi_j(t)$$

$\{n_2(t)\}$: Random process $N_2(t)$.

$$\longrightarrow \quad N(t) = N_1(t) + N_2(t)$$

$N(t) \longleftrightarrow \underline{N} = [N_1 \, N_2 \cdots N_m]$ and $N_2(t)$: "orthogonal"

Claim: $\underline{N} \sim$ iid Normal $N(0, \frac{N_0}{2})$

$\underline{\text{Pf}}$: $N_j = \int_{-\infty}^{\infty} \underbrace{N(t)}_{RP} \phi_j(t) \, dt \longrightarrow$ RV, Gaussian

$$E[N_i] = \int_{-\infty}^{\infty} \underbrace{E[N(t)]}_{0} \phi_i(t)\, dt = 0$$

$$\mathrm{Cov}(N_i, N_j) = E[N_i N_j] = E\left[\int_{-\infty}^{\infty} N(t)\phi_i(t)\,dt \int_{-\infty}^{\infty} N(\tau)\phi_j(\tau)\,d\tau\right]$$

$$= \int\int \underbrace{E[N(t)N(\tau)]}_{\frac{N_0}{2}\delta(t-\tau)} \phi_i(t)\phi_j(\tau)\,dt\,d\tau$$

$$= \frac{N_0}{2}\delta_{ij}$$

Claim: $N_2(t)$ and N_i are independent $\forall t$.

Pf: $E[N_i N_2(t)] = 0$

Received Signal:

$$Y(t) = X(t) + N(t)$$

$$= \sum_{j=1}^{m} X_j \phi_j(t) + \sum_{j=1}^{m} N_j \phi_j(t) + N_2(t)$$

$$= \underbrace{\sum_{j=1}^{m} (X_j + N_j) \phi_j(t)} + N_2(t)$$

independent

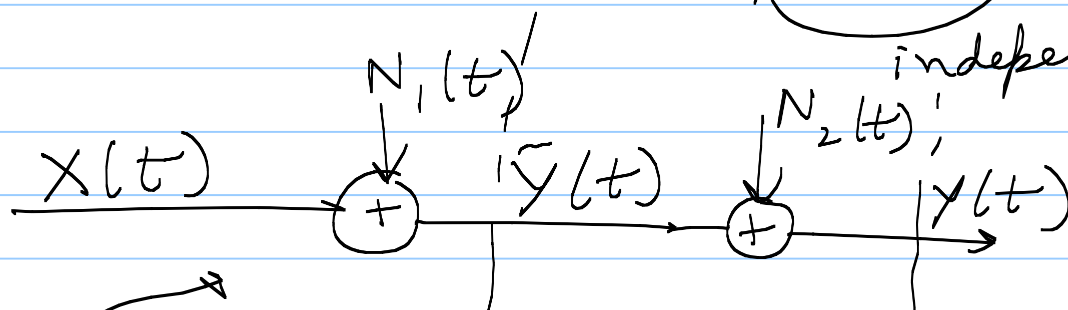
Let $Y_j = X_j + N_j$

$$\underline{Y} = [Y_1 \ Y_2 \ \ldots \ Y_m]$$

$$\tilde{y}(t) = \sum_{j=1}^{m} Y_j \phi_j(t)$$

$$Y(t) = \tilde{Y}(t) + N_2(t)$$

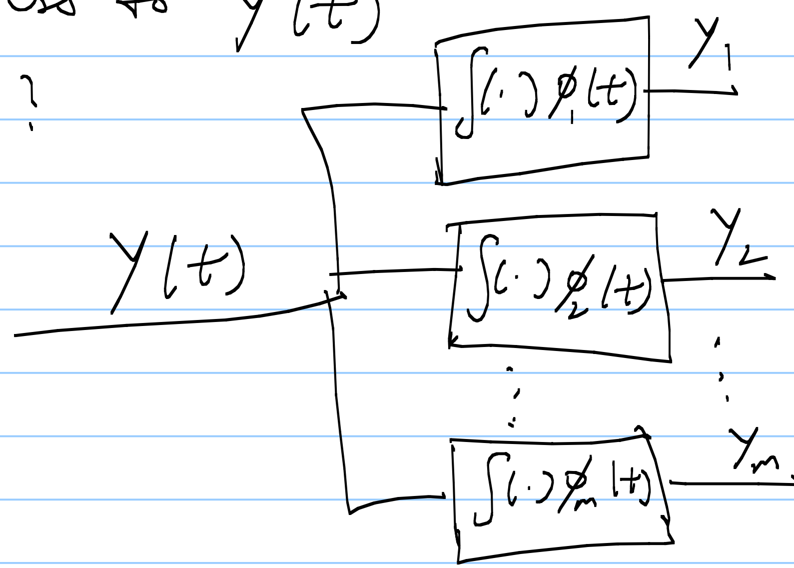
independent



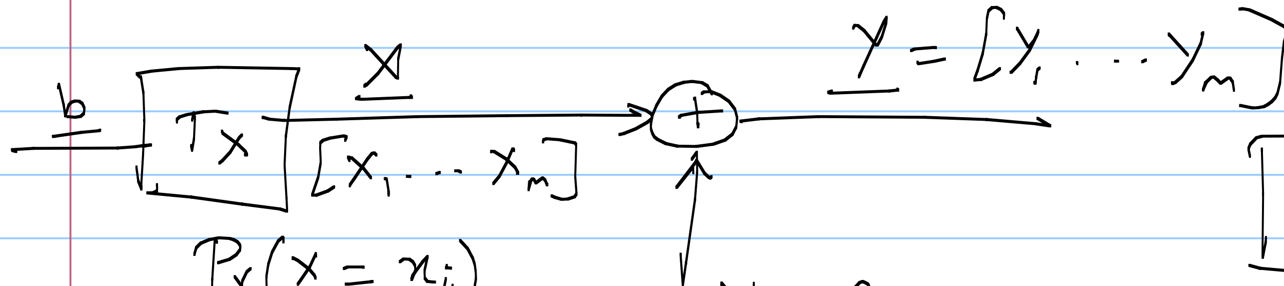
receiver has access to $\tilde{Y}(t)$

why?

→ Given $\tilde{Y}(t)$, $X(t)$ & $Y(t)$ are independent.



Vector Model



$$\Pr(\underline{X} = \underline{x}_i) = \frac{1}{2^N}$$

$$\underline{N} = [N_1 \cdots N_m]$$

iid $N(0, \frac{N_0}{2})$

$$\boxed{Y_i = X_i + N_i}$$

Scalar model