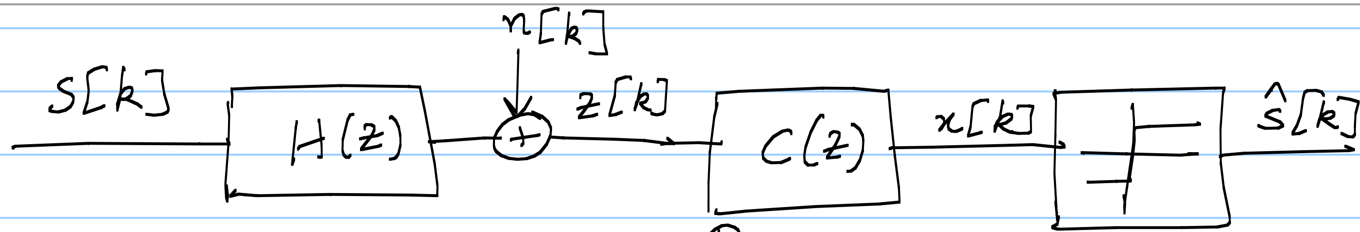
Note Title 9/30/2008

Lecture 32



$$C(z) = \sum_{m=-P}^{P} c_m z^{-m}$$

Order: $N = 2P+1$

$$\underline{c} = \begin{bmatrix} c[-P] \\ \vdots \\ c[0] \\ \vdots \\ c[P] \end{bmatrix}$$

$$e[k] = x[k] - s[k]$$

$$MSE = E\left[\,|e[k]|^2\right]$$

$$\underline{z}_k = \begin{bmatrix} z[k+P] \\ \vdots \\ z[k] \\ \vdots \\ z[k-P] \end{bmatrix}$$

$$x[k] = \underline{c}^T \underline{z}_k$$

$$MSE = E\left[|e[k]|^2\right] = \underbrace{E\left[|s[k]|^2\right]}_{E_s} + \underline{c}^{*T} \underbrace{E\left[\underline{z}_k^{*} \underline{z}_k^{T}\right]}_{N \times N} \underline{c}$$

$$-2\,\mathrm{Re}\left\{\underline{c}^{*T} \underbrace{E\left[s[k]\,\underline{z}_k^{*}\right]}_{N \times 1}\right\}$$

$$\underline{d} = E\begin{bmatrix} s[k]\, z^{*}[k+P] \\ \vdots \\ s[k]\, z^{*}[k] \\ \vdots \\ s[k]\, z^{*}[k-P] \end{bmatrix} = \;?$$

independent of k.

$$\phi = E\left[\underline{z}_k^* \underline{z}_k^T\right] = E\left[\begin{bmatrix} z^*[k+p] \\ \vdots \\ z^*[k] \\ \vdots \\ z^*[k-p] \end{bmatrix}\begin{bmatrix} z[k+p] & \cdots & z[k] & \cdots & z[k-p] \end{bmatrix}\right]$$

$$R_z[m] = E\left[z[k+m]\, z^*[k]\right] \overset{DTFT}{\longleftrightarrow} S(e^{j\theta})$$

real, non-negative

$$R_z^*[m] = R_z[-m]$$

$$\phi = \begin{bmatrix} R_z[0] & R_z[-1] & \cdots & R_z[-2p] \\ R_z[1] & R_z[0] & \cdots & R_z[-(2p-1)] \\ \vdots & & \ddots & \vdots \\ R_z[2p] & R_z[2p-1] & \cdots & R_z[0] \end{bmatrix}$$

$\rightarrow$ $[\phi]_{ij} = R_z[i-j]$ "Toeplitz".

$0 \leq i, j \leq 2p$

$\rightarrow$ ϕ : Hermitian symmetric $\phi^{*T} = \phi$.

$\rightarrow$ ϕ : Positive definite

$$\underline{x}^{*T} \phi \underline{x} \geq 0 \quad \forall \ \underline{x}$$

$$MSE = E\left[|s[k]|^2\right] - 2\,\mathrm{Re}\left\{\underline{c}^{*T}\underline{\alpha}\right\} + \underline{c}^{*T}\phi\,\underline{c}$$

$\updownarrow$ equal

Rewrite,

$$MSE = E\left[|s[k]|^2\right] - \underline{\alpha}^{*T}\phi^{-1}\underline{\alpha} + \underbrace{(\phi^{-1}\underline{\alpha}-\underline{c})^{*T}\phi(\phi^{-1}\underline{\alpha}-\underline{c})}_{\geq 0}$$

$$\boxed{\begin{aligned} \underline{c}_{opt} &= \phi^{-1}\underline{\alpha} \\ MMSE &= E_s - \underline{\alpha}^{*T}\underline{c}_{opt} \end{aligned}}$$

$\longrightarrow$ $MSE = f(\underline{c})$ $\qquad \nabla_{\underline{c}} MSE = 2\phi\,\underline{c} - 2\underline{\alpha}$

$$\underline{c}_{opt} = \phi^{-1}\underline{\alpha}$$

Deriving MMSE-LE of order $N = 2P+1$

$\Updownarrow$

Solving linear system of equations

$$\underset{N \times N}{\underline{\phi}}\ \underset{N \times 1}{\underline{c}} = \underset{N \times 1}{\underline{\alpha}}$$

$$z[k] = s[k] * h[k] + n[k]$$

$$\underline{\alpha} = \begin{bmatrix} E\left[s[k] z^*[k+P]\right] \\ \vdots \\ E\left[s[k] z^*[k-P]\right] \end{bmatrix}$$

$$[\underline{\phi}]_{ij} = R_z[i-j]$$

$$0 \le i, j \le 2P$$

MSE Gradient Algorithm (known: ϕ, $\underline{\alpha}$)

iterative: $\underline{c}_0, \underline{c}_1, \underline{c}_2, \ldots, \underline{c}_j, \underline{c}_{j+1}, \ldots \rightarrow \underline{c}_{opt}$

$$\underline{c}_{j+1} = \underline{c}_j - \underbrace{\frac{\beta}{2}}_{\text{step size}} \underbrace{\nabla_{\underline{c}_j} E\left[|e[k]|^2\right]}_{\text{gradient}}$$

$$\underline{c}_{j+1} = \underline{c}_j + \beta(\underline{\alpha} - \phi\, \underline{c}_j)$$

$$= (I - \beta\phi)\, \underline{c}_j + \beta\, \underline{\alpha}$$

$$\underline{q}_{j+1} = \underline{c}_{j+1} - \underbrace{\underline{c}_{opt}}_{\phi^{-1}\underline{\alpha}} = (I - \beta\phi)\, \underline{q}_j$$

$$\underline{q}_{j+1} = \underbrace{(I - \beta\phi)^{j+1}}_{\downarrow \; \underline{0}_{N\times N}} \underline{q}_0$$

ϕ: Hermitian symmetric.

"Spectral decomposition"

$$\phi = \sum_{i=1}^{n} \underbrace{\lambda_i}_{\text{eigenvalue of } \phi} \underbrace{\underline{v}_i \, \underline{v}_i^{*T}}$$

$$\underline{v}_i^{*T} \underline{v}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$\lambda_{max} = \lambda_1 \geq \lambda_2 \geq \cdots\cdots \geq \lambda_n = \lambda_{min}$$

$$(I - \beta\phi)^j = \sum_{i=1}^{n} (1 - \beta\lambda_i)^j \underline{v}_i \, \underline{v}_i^{*T} \xrightarrow{?} 0$$

MSE Gradient $\longrightarrow \underline{c}_{opt}$

if $\quad |1-\beta\lambda_i| < 1 \qquad i=1,2,\ldots,n$

$$-1 < 1-\beta\lambda_{max} < 1$$

Cm)

$$0 < \beta < \frac{2}{\lambda_{max}}$$

<u>Fastest</u>

$$\underline{c}_j \longrightarrow \underline{c}_{opt} \quad \Rightarrow \quad \beta_{opt} = \frac{2}{\lambda_{min} + \lambda_{max}}$$

<u>Fastest</u>

$$MSE_j \longrightarrow 0 \quad \Rightarrow \quad \boxed{\beta = \frac{1}{\lambda_{max}}}$$