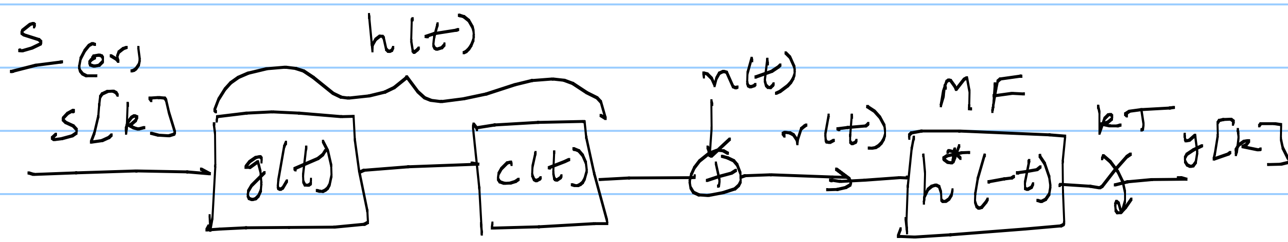
Lecture 19

Spectral factorization:

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$R_{xx}[m] \xleftrightarrow{DTFT} |X(e^{j\omega})|^2$$



$$J_{\underline{a}} = \text{distance}\left(r(t), \sum_{k=0}^{L-1} a[k]h(t-kT)\right)$$

$$y[k] = s[k] * p_h[k] + n'[k]$$

PSD $S_h(e^{j2\pi fT})$

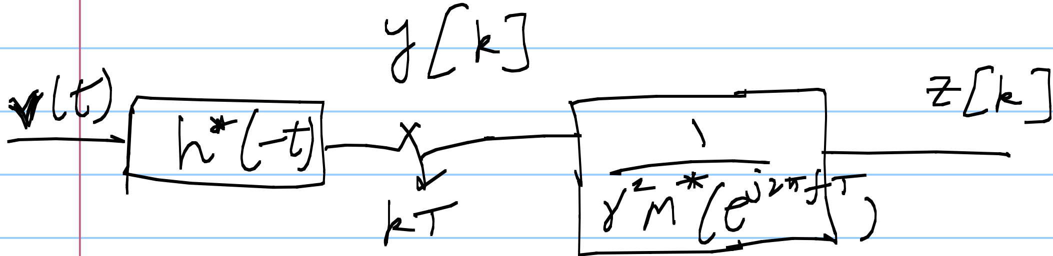
$$S_h(e^{j2\pi fT}) = \gamma^2 M(e^{j2\pi fT}) M^*(e^{j2\pi fT})$$

$$\updownarrow$$

$m[k]$: loosely minimum phase

$$S_h(z) = \gamma^2 M(z) M^*\left(\frac{1}{z^*}\right)$$

$$\rho_h[k] = \gamma^2 m[k] * m^*[-k]$$



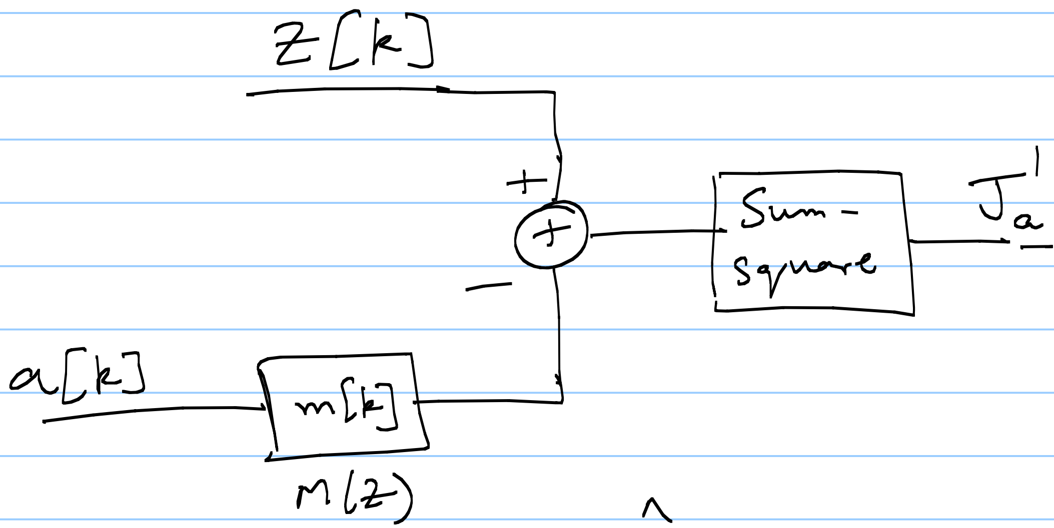
J_a

$$\Updownarrow$$

$$J_a' = \sum_{k=0}^{\infty} \left| z[k] - a[k] * m[k] \right|^2$$

$$z[k] = s[k] * m[k] + n[k]$$

white, $N\left(0, \frac{N_0}{2\gamma^2}\right)$



$$\hat{\underline{s}} = \arg\min_{\underline{a} \in \mathcal{X}^L} J'_{\underline{a}}$$

$$\downarrow$$

$$|\mathcal{X}|^L$$

Orthonormal Basis

Set of signals (at Rx!!)

$\underline{a} \in \mathcal{X}^L$ $\qquad \sum_{k=0}^{L-1} a[k]\, h(t-kT)$

$$h(t) \overset{FT}{\longleftrightarrow} H(f)$$

$$h(t) * h^*(-t) \overset{FT}{\longleftrightarrow} |H(f)|^2$$

$$h(t) * h^*(-t)\Big|_{kT} \overset{DTFT}{\longleftrightarrow} \frac{1}{T}\sum_{m=-\infty}^{\infty} \left|H\left(f-\frac{m}{T}\right)\right|^2$$

$$= S_h\left(e^{j2\pi fT}\right)$$

"Spectral Factorization" $S_h\left(e^{j2\pi fT}\right) = \gamma^2\, M\left(e^{j2\pi fT}\right) M^*\left(e^{j2\pi fT}\right)$

$$\phi(t) \xleftrightarrow{FT} \Phi(f) = \frac{H(f)}{\sqrt{M(e^{j2\pi fT})}}$$

Claim:

Linear space $S = \text{Span}\{h(t-kT)\}$.

$$\sum_{k=\cdots}^{\cdots} s[k]\, h(t-kT)$$

$\{\phi(t-kT)\}$: orthonormal basis for S.

Pf:

$$\int_{-\infty}^{\infty} \phi(t)\phi^*(t-kT)\,dt = \phi(t) * \phi^*(-t)\Big|_{kT}$$

$$\Big\updownarrow \text{DTFT}$$

$$\frac{1}{T}\sum_{m=-\infty}^{\infty} \left|\Phi\left(f-\frac{m}{T}\right)\right|^2$$

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} \frac{\left|H\left(f-\frac{m}{T}\right)\right|^2}{\gamma^2 \left|M\left(e^{j2\pi\left(f-\frac{m}{T}\right)T}\right)\right|^2}$$

$$= \frac{1}{\gamma^2 \left|M\left(e^{j2\pi f T}\right)\right|^2} \cdot \frac{1}{T} \sum_{m=-\infty}^{\infty} \left|H\left(f-\frac{m}{T}\right)\right|^2$$

$$S_h\left(e^{j2\pi f T}\right)$$

$$= 1$$

Spanning:

$$s(t) = \sum s[k]\, h(t-kT)$$

To show: $\exists\, b[k]$ s.t. $s(t) = \sum b[k]\, \phi(t-kT)$

$$\xrightarrow[\sum s[k]\delta(t-kT)]{s[k]} \boxed{h(t)} \xrightarrow{s(t)}$$

$$\phi(f) = \frac{H(f)}{\gamma M(e^{j2\pi fT})}$$

$$\updownarrow$$

$$\gamma\, \phi(t) * \sum m[k]\delta(t-kT) = h(t)$$

$$\xrightarrow{s[k]} \boxed{\gamma m[k]} \xrightarrow{b[k]} \boxed{\phi(t)} \xrightarrow{s(t)}$$

$$s(t) = \sum b[k]\,\phi(t-kT)$$

$$b[k] = \gamma\, s[k] * m[k]$$

$$r(t) \rightarrow \boxed{\phi^*(-t)} \xrightarrow{kT} z'[k] \rightarrow \boxed{\text{Detector}} \rightarrow \hat{s}$$

$$z'[k] = \gamma\left(s[k] * m[k]\right) + n[k]$$

$n[k]$: WGN, $\mathcal{N}\left(0, \frac{N_0}{2}\right)$

$$\phi^*(-t) \xleftrightarrow{FT} \frac{H^*(f)}{\underbrace{\gamma M^*\left(e^{j2\pi fT}\right)}}$$