Assignment 1

Basics, Source coding, RLL codes

Date Assigned: Feb 9

1. *Example of Joint entropy.* Let $p(x, y)$ given by

$X \setminus Y$	0	1
0	1/3	1/3
1	0	1/3

   Find

   (a) $H(X)$, $H(Y)$

   (b) $H(X|Y)$, $H(Y|X)$

   (c) $H(X, Y)$, $I(X; Y)$

2. Let X, Y, and Z be discrete random variables with a joint distribution. Prove the validity of the following inequalities and if true find conditions for equality:

   (a) $I(X, Y; Z) \geq I(X; Z)$.

   (b) $H(X, Y|Z) \geq H(X|Z)$.

   (c) $H(X, Y; Z) \geq I(X; Z)$.

   (d) $I(X; Z|Y) \geq I(Z; Y|X) - I(Z; Y) + I(X; Z)$.

   (e) $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$.

3. *A measure of correlation.* Let X_1 and X_2 be identically distributed, but not necessarily independent. Let

$$\rho = 1 - \frac{H(X_2|X_1)}{H(X_1)}$$

   (a) Show $\rho = I(X_1; X_2)/H(X_1)$.

   (b) Show $0 \leq \rho \leq 1$

   (c) When is $\rho = 0$? When is $\rho = 1$?

4. Let X, Y and Z be binary random variables so that the eight elements in the joint {X,Y,Z} ensemble can be taken as the vertices of a unit cube.

   (a) Find a joint probability assignment $p(x, y, z)$ such that $I(X; Y) = 0$ and $I(X; Y|Z) = 1$ bit.

   (b) Find the joint probability assignment $p(x, y, z)$ such that $I(X; Y) = 1$ bit and $I(X; Y|Z) = 0$.

   The point of the problem is that no general inequality exists between $I(X; Y)$ and $I(X; Y|Z)$.

5. *Conditional Mutual Information.* Consider a sequence of n binary random variables X_1, X_2, X_3, ..., X_n. Each sequence with an even number of 1's has the probability $2^{-(n-1)}$ and each sequence with odd number of 1's has the probability 0. Find the mutual informations $I(X_1; X_2)$, $I(X_2; X_3|X_1)$, ..., $I(X_{n-1}; X_n|X_1, ..., X_{n-2})$.

6. *The AEP and source coding.* A discrete memoryless source emits a sequence of statistically independent binary digits with probabilities $p(1) = 0.005$ and $p(0) = 0.995$. The digits are taken 100 at a time and a binary codeword is provided for every sequences of 100 digits containing three or fewer ones.

   (a) Assuming that all codewords are the same length, find the minimum length required to provide codewords for all sequence with three or fewer ones.

(b) Calculate the probability of observing a source sequence for which no codeword has been assigned.

7. *Entropy rates of Markov chains.*

(a) Find the entropy rate of the two-state Markov chain with transition matrix

$$P = \begin{bmatrix} 1-p_{01} & p_{01} \\ p_{10} & 1-p_{10} \end{bmatrix}$$

(b) What values of p_{01} and p_{10} maximize the rate of Part (a)?

(c) Find the entropy rate of the two-state Markov chain with transition matrix

$$P = \begin{bmatrix} 1-p & p \\ 1 & 0 \end{bmatrix}$$

(d) Find the maximum value of the entropy rate of the Markov chain in Part (c). We expect that the maximizing value of p should be less than $1/2$, since the 0 state permits more information to be generated than the 1 state.

8. *Initial conditions.* For a Markov chain $\{X_n\}$, show that

$$H(X_0|X_n) \geq H(X_0|X_{n-1})$$

Thus initial conditions X_0 is become more difficult to recover as the future X_n unfolds.

9. *Pairwise independence.* Let X_1, X_2, $\cdots$, X_{n-1} be *i.i.d* random variables taking values in {0,1}, with $\Pr\{X_i = 1\} = 1/2$. Let $X_n = 1$ if $\sum_{i=1}^{n-1} X_i$ is odd and $X_n = 0$ otherwise. Let $n \geq 3$.

(a) Show that X_i and X_j are independent for $i \neq j$, $i, j \in \{1, 2, ..., n\}$.

(b) Find $H(X_i, X_j)$ for $i \neq j$.

(c) Find $H(X_1, X_2, ..., X_n)$. Is this equal to $nH(X_1)$?

10. *Huffman coding.* Consider the random variable

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.49 & 0.26 & 0.12 & .04 & 0.04 & 0.03 & 0.02 \end{pmatrix}$$

(a) Find a binary Huffmann code for X.

(b) Find the expected codelength for this encoding.

(c) Find a ternary Huffmann code for X.

11. *More Huffman codes.* Find the binary Huffman code for the source with probabilities (1/3, 1/5, 1/5, 2/15, 2/15). Argue that this code is also optimal for the source with probabilities (1/5, 1/5, 1/5, 1/5, 1/5).

12. *Bad codes.* Which of these codes cannot be Huffman code for any probability assignment?

(a) {0,10,11}

(b) {00,01,10,110}

(c) {01,10}

13. Consider a random variable X which takes on four values with probabilities (1/3,1/3,1/4,1/12).

(a) Construct a Huffman code for this random variable

(b) Show that there exist two differant sets of optimal length for the codewords, namely, show that codeword length assignments (1,2,3,3) and (2,2,2,2) are both optimal.

14. *Huffman code.* Find the (a) binary and (b) ternary Huffman codes for the random variable X with probabilities

$$p = \left(\frac{1}{21}, \frac{2}{21}, \frac{3}{21}, \frac{4}{21}, \frac{5}{21}, \frac{6}{21}\right).$$

(c) Calculate $L = \sum p_i l_i$ in each case.

15. *Classes of codes.* Consider the code {0,01}
    (a) Is it instantaneous?
    (b) Is it uniquely decodable?
    (c) Is it nonsingular?
16. A source has an alphabet of 4 letters. The probabilities of the letters and two possible sets of binary codeword for the source are given below:

Letter	Prob.	Code I	Code II
a_1	0.4	1	1
a_2	0.3	01	10
a_3	0.2	001	100
a_4	0.1	000	1000

    For each code, answer the following questions (no proofs or numerical answers are required).
    (a) Does the code satisfy the prefix condition?
    (b) Is the code uniquely decodable?
    (c) What is the mutual information provided about the event that the source letter is a_1 by the event that the first letter of the codeword is 1?
    (d) What is the average mutual information provided about the source letter by the specification of the first letter of the codeword? Give a heuristic description of the purpose of the first letter in the codewords of Code II.
17. A discrete memoryless source with alphabet $A = \{a, b, c\}$ emits the following string.

    b c c a c b c c c c c c c c c c c a c c c a

    Using the Lempel-Ziv algorithm, encode this sequence and find the code dictionary and transmitted sequence.
18. A source with alphabet $A = \{a, b, c\}$ is encoded using the Lempel-Ziv algorithm. The transmitted codeword sequence is

$$2, 3, 3, 1, 3, 4, 5, 10, 11, 6, 10.$$

    Construct the dictionary and decode the sequence.